Chapter 2

Optical bistability in one-dimensional nonlinear photonic band gap structures

2.1 Introduction

The propagation of waves through periodic dielectric structures, called photonic band gap structures (PBG), has been extensively studied in recent years (see e.g. Ref. [11], [17] and [18]). An essential property of these structures is the existence of a frequency band gap in which light propagation is forbidden. This is analogous to the electronic band gaps in semiconductor crystals. In such a crystal, a moving electron experiences a periodic potential produced by the atomic lattice, which produces a gap in the electronic energy band. This gap splits the energy band into two parts: the lower energy band is called the valence band and the high energy band is the conduction band. The optical analogy is the photonic crystal where the periodic potential is due to a lattice of different macroscopic dielectric media. However, when a defect layer is introduced into an otherwise strictly periodic PBG structure, it can create donor or acceptor modes in the band gap [26]. Similar to the case of an electron being localized around a defect crystal, there is a large field enhancement in the optical defect structure.

When Kerr nonlinearity is introduced in the PBG structures (the effective refractive index now depends on the field intensity) it will alter the transmission spectrum including the position of the band-edges. This dynamic shifting of the band-edges can produce optical bistability phenomena; see e.g. Ref. [16], [20] and [23]. In Ref. [20] the author introduced an optical switch in a nonlinear finite grating using two light
pulses at different frequencies, a probe beam near the band-edge and a strong pump beam in the middle of the band gap to alter the index of refraction of the structure. The pump beam is used to control the position of the band-edge of the structure so as to transmit or reflect the probe beam. A drawback of this procedure was that the pump beam did not penetrate far into the structure and therefore did not change the effective index of refraction significantly. To increase the field penetration but prevent the transmission, Tran has introduced a defect layer in the grating structure and has shown that the defect structure can indeed improve the performance of these switching devices, see Ref. [21]. Since the switching mechanism consists of two pulses at different frequencies, Tran [20] and [21] used the nonlinear finite difference time domain (NFDTD) method to go beyond the slowly varying envelope approximation and to avoid the complexity of the problem due to the interaction between two frequencies.

A different approach for an all-optical switch is based on the self-switching arrangement where the output depends nonlinearly on some characteristics of the input light, e.g. power. An example is optical bistability which can be realized among others in a nonlinear periodic structure. In this case the input light may be monochromatic. While the threshold value of bistability needed by a periodic structure is relatively high, He and Cada [9] proposed a combined structure which is composed of a distributed feedback structure and a phase-matching layer placed in a Fabry-Perot cavity in order to reduce the threshold significantly. Recently, Lidorikis et al. [12] and Wang et al. [23] found that when a single nonlinear defect layer is introduced into a linear periodic structure, the threshold value of the bistability is compatible with that of the structure proposed by He and Cada [9]. Throughout this chapter we study a bistability that uses a monochromatic light input in finite periodic structures without and with a defect. For the defect structure, the nonlinearity will not only be introduced in the defect layer but also in the higher and/or lower index layers. Because we deal with a single frequency, the Maxwell’s equations are reduced to a nonlinear Helmholtz (NLH) equation. Furthermore, the numerical calculations in the frequency domain are generally more efficient than those in time domain.

Since 30 years ago, a lot of efforts have been devoted to study the phenomenon of bistability in a periodic structure. A number of authors, e.g. Marburger and Felber [15], Winful et al. [25], Danckaert et al. [6] and [7] proposed an analytical formalism for this problem. All these formalisms are derived within three basic approximations, i.e. the slowly varying envelope approximation (SVEA), the approximation of nonlinear terms that appear in the interface conditions, and the omission of spatial third harmonics generated in the structure. Treatments that make use of the full nonlinear interface conditions in the nonlinear transfer matrix were given by Agarwal and Dutta Gupta [1] and Dutta Gupta and Agarwal [8].

Another approach to solve the nonlinear wave equation was proposed by Chen and Mills [4] and [5]. In this approach the NLH equation is transformed into a phase-amplitude equation. By combining with energy conservation the phase-amplitude
equation is written in integral form. The integral equation together with the continuity conditions at the interfaces are solved numerically. Recently this method has been implemented by Lidorikis et al. [12] to investigate the localized mode solution for a single nonlinear layer sandwiched between two linear periodic structures.

A semi-analytic method has been proposed by Wang et al. [23] to study the optical bistability in a linear structure with a single nonlinear defect layer in the center. The transfer matrix method is used for the linear part and a finite difference method is implemented for the nonlinear layer. The left and right linear parts and the nonlinear layer are linked using appropriate interface conditions.

In this chapter we discuss a finite element method (FEM) to study the nonlinear optical response of one-dimensional (1D) finite grating structures. We will directly implement the exact NLH equation and transparent-influx boundary conditions (TIBC) which will be derived in section 2.2. In section 2.3 the NLH equation together with TIBC is transformed into a variational numerical scheme. For the linear scheme we improve the standard FEM to get a fourth order accurate scheme that maintains the symmetric-tridiagonal structure of the finite element matrix. For the full nonlinear equation, we implement the improved FE scheme for the linear part and a standard FEM for the nonlinear part. The resulting nonlinear system of equations will be solved using two different input parameters, i.e. either the amplitude of the incident wave or that of the transmitted wave. In the subsequent sections we apply our numerical scheme to study the optical response of both linear and nonlinear grating structures. Finally we end this chapter with conclusions and remarks.

2.2 Wave equation and its boundary conditions

We consider the propagation of optical electromagnetic field through one-dimensional, periodic, dispersionless and lossless stratified dielectric media with a Kerr nonlinearity. The electric and the magnetic fields have the form,

$$
\mathbf{E}(z,t) = [0, E_y(z,t), 0],
$$

$$
\mathbf{H}(z,t) = [H_x(z,t), 0, 0].
$$

Assuming that the media are isotropic, the polarization $\mathbf{P}$ is parallel to $\mathbf{E}$:

$$
\mathbf{P}(z,t) = [0, P_y(z,t), 0].
$$

Maxwell’s equations then reduce to the wave equation

$$
\frac{\partial^2 E_y(z,t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E_y(z,t)}{\partial t^2} = \mu_0 \frac{\partial^2 P_y(z,t)}{\partial t^2}.
$$

For a material with a Kerr nonlinearity, the polarization $P_y$ is given by (see He and Liu [10])

$$
P_y(z,t) = \varepsilon_0 \left\{ \chi^{(1)}(z) E_y(z,t) + \chi^{(3)}(z) |E_y(z,t)|^2 E_y(z,t) \right\},
$$
where $1 + \chi^{(1)}(z) \equiv \varepsilon_r(z) \equiv n^2(z)$ describes the linear dielectric constant, $n$ is the refractive index and $\chi^{(3)}$ is the third-order nonlinear susceptibility. If excited by an input beam with a single frequency $\omega$, the polarization (2.5) does not lead to the generation of high order harmonics in the Kerr medium. Hence we can look for stationary solutions of the form

$$E_y(z, t) = \exp(i\omega t) E(z).$$  \hspace{1cm} (2.6)

The harmonic time dependence leads to the scalar NLH equation

$$\frac{d^2 E(z)}{dz^2} + k^2 \left(n^2(z) + \chi^{(3)}(z)|E(z)|^2\right) E(z) = 0,$$  \hspace{1cm} (2.7)

where $k^2 = \omega^2/c^2$ with $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the speed of light. Notice that equation (2.7) holds for arbitrary $n(z)$ and $\chi^{(3)}(z)$. In many applications, we deal with the NLH (2.7) in the presence of an incident wave from either a linear or nonlinear homogeneous medium to a scattering structure (e.g. the linear or nonlinear periodic structure) that is followed by a linear or nonlinear homogeneous region. Here we assume that the medium outside the scattering structure is linear and homogeneous with refractive index $n_0$. In order to solve this problem numerically, we have to limit the original unbounded domain to a finite computational domain. In doing so, we have to provide boundary conditions that can transmit completely the incident wave and simultaneously be transparent for all outgoing waves. These boundary conditions are called transparent-influx boundary conditions (TIBC). Such boundary conditions will be derived as follows. Outside the scattering structure, the Helmholtz equation is linear and has a constant refractive index $n_0$. Therefore this equation can be factorized as

$$\left(\frac{d}{dz} - ikn_0\right)\left(\frac{d}{dz} + ikn_0\right) E = 0,$$  \hspace{1cm} (2.8)

yielding the following boundary conditions:

$$\frac{dE}{dz} - ikn_0 E = -2 i k n_0 A_{\text{inc}}, \quad z = z_{\text{min}}$$  \hspace{1cm} (2.9)

$$\frac{dE}{dz} + ikn_0 E = 0, \quad z = z_{\text{max}}.$$  \hspace{1cm} (2.10)

The first boundary condition (2.9) is an influx condition for an incident wave with wavenumber $k$ and constant amplitude $A_{\text{inc}}$ and is simultaneously transparent for the back scattered field. The second condition (2.10) is a transparent boundary condition for the right-traveling wave.

Throughout this chapter we deal with a one-dimensional finite quarter-wavelength stack which is composed of alternating layers which have a high refractive index $n_H$ (denoted as $H$ layer) and a low refractive index $n_L$ (denoted as $L$ layer). Figure 2.1 illustrates the geometry. The thicknesses for the two kinds of layers are such that
Figure 2.1: Schematic view of the 1D PBG structures considered in this paper which are composed of \( N_1 \) HL layers and \( N_2 \) LH layers separated by a defect layer with thickness \( L_d = M \times (\lambda_0/4n_d) \). The thicknesses of layers \( H \) and \( L \) are respectively \( d_H = \lambda_0/4n_H \) and \( d_L = \lambda_0/4n_L \). The medium outside the structure is linear homogeneous with refractive index \( n_0 \). For the numerical calculations we introduce a transparent-influx boundary condition (TIBC) and transparent boundary condition (TBC) in the left and right hand side of the structure.

\[ d_L = \lambda_0/4n_L \text{ and } d_H = \lambda_0/4n_H, \text{ where } \lambda_0 \text{ is the free-space design wavelength. We assume that the front and back media have refractive index } n_0. \text{ It is also assumed that the high-index layer shows a positive Kerr nonlinearity. The defect structure can be obtained by simply perturbing the thickness of any layer (which is then called a defect layer, denoted by } D) \text{ or by changing the refractive index of the defect layer. For simplicity we denote this structure } \text{(HL)}^{N_1} (D)^M (LH)^{N_2}, \text{ where } N_1 \text{ and } N_2 \text{ are respectively the number of layer periods in the left and right of the defect layer. Here } M \text{ determines the thickness of the defect layer } L_d \text{ such that } L_d = M \times (\lambda_0/4n_d) \text{ where } n_d \text{ is the refractive index of the defect layer.}\]

2.3 Numerical Method

2.3.1 Linear scheme

In this section we discuss a numerical method to solve Equation (2.7) based on a variational method. We first concentrate to only the linear part, i.e. where \( \chi^{(3)} = 0 \) and consider the functional

\[
\mathcal{F}_{Lin}(E) = -\frac{1}{2} \int_0^L \left( \left| \frac{dE}{dz} \right|^2 - k^2 n^2 |E|^2 \right) dz
\]
where \( l < z_{\text{min}} \) and \( L > z_{\text{max}} \). This functional can also be written as

\[
\mathcal{F}_{\text{Lin}} (E) = \mathcal{F}_1 (E) + \mathcal{F}_2 (E) + \mathcal{F}_3 (E)
\]  

(2.12)

where

\[
\mathcal{F}_1 (E) = -\frac{1}{2} \int_{l}^{z_{\text{min}}} \left( \frac{d|E|^2}{dz} - k^2 n_0^2 |E|^2 \right) dz,
\]

\[
\mathcal{F}_2 (E) = -\frac{1}{2} \int_{z_{\text{min}}}^{z_{\text{max}}} \left( \frac{d|E|^2}{dz} - k^2 n_0^2 |E|^2 \right) dz,
\]

\[
\mathcal{F}_3 (E) = -\frac{1}{2} \int_{z_{\text{max}}}^{L} \left( \frac{d|E|^2}{dz} - k^2 n_0^2 |E|^2 \right) dz.
\]

As it is assumed that the medium outside the grating structure is linear homogeneous with refractive index \( n_0 \), without loss of generality, the solution of equation (2.7) can be written as

\[
E (z) = \begin{cases} 
A_{\text{inc}} \exp (-ikn_0 (z - z_{\text{min}})) + A_{\text{ref}} \exp (ikn_0 (z - z_{\text{min}})) & , \ z \in [l, z_{\text{min}}]; \\
E (z) & , \ z \in [z_{\text{min}}, z_{\text{max}}]; \\
A_{\text{tr}} \exp (-ikn_0 (z - z_{\text{max}})) & , \ z \in [z_{\text{max}}, L]; 
\end{cases}
\]

(2.13)

where \( A_{\text{inc}} \) and \( A_{\text{ref}} \) are respectively the amplitudes of the incident and the reflected waves and \( A_{\text{tr}} \) is the transmitted wave amplitude. By substituting Equation (2.13) into the functional (2.12), one can check that

\[
\mathcal{F}_1 = -\frac{kn_0}{\sin \left( kn_0 l \right)} \left[ \cos \left( kn_0 l \right) \left( |E_0|^2 + |E_l|^2 \right) - (E_0 E_l^* + E_l E_0^*) \right]
\]

\[
\mathcal{F}_3 = -\frac{kn_0}{\sin \left( kn_0 L \right)} \left[ \cos \left( kn_0 L \right) \left( |E_{M_0}|^2 + |E_{L}|^2 \right) - (E_{M_0} E_{L}^* + E_{L} E_{M_0}^*) \right]
\]

where \( l = l - z_{\text{min}} \), \( L = z_{\text{max}} - L \), \( E_0 = E (z_{\text{min}}) \), \( E_l = E (l) \), \( E_{M_0} = E (z_{\text{max}}) = A_{\text{tr}} \) and \( E_{L} = E (L) \); and \( E^* \) stands for the complex conjugate of \( E \). Then one can show that if the variational derivative of functional (2.12) vanishes, \( \delta_{E} \mathcal{F}_{\text{Lin}} = 0 \), then the field \( E (z) \) satisfies the linear part of the NLH equation (2.7) and its boundary conditions (2.9) and (2.10). To derive the numerical scheme, we approximate the functional \( \mathcal{F}_2 (E) \) by writing the function \( E (z) \) as a linear combination of a linear “hat” basis function \( \{ \varphi_m (z) \}_{0}^{M_0} \):

\[
E (z) \cong \sum_{j=0}^{M_0} \hat{E}_j \varphi_j (z)
\]

(2.14)

such that

\[
\mathcal{F}_2 (E) \cong \tilde{\mathcal{F}}_2 \left( \hat{E} \right)
\]

(2.15)
where $\tilde{E} = (\tilde{E}_0, \tilde{E}_1, \ldots, \tilde{E}_{M_0})^T$. Here we assume that the interval $[z_{\text{min}}, z_{\text{max}}]$ is divided into $M_0$ subintervals of equal length $h = (z_{\text{max}} - z_{\text{min}})/M_0$ by choosing the nodal points $z_j = z_{\text{min}} + jh$ for $j = 0, 1, \ldots, M_0$. $\tilde{E}_j$ is the approximation of $E(z)$ at $z = z_j$. The condition $\delta E \mathbf{F}_{\text{Lin}} = 0$ therefore corresponds to $\nabla \mathbf{F}_{\text{Lin}} (\tilde{E}) = 0$ which leads to the finite element scheme (by assuming that each discontinuity of the structure coincides with a grid point)

$$
\left( \frac{1}{h} P + \frac{1}{6} h k^2 Q \right) \tilde{E} = v,
$$

(2.16)

where

$$
P = \begin{pmatrix}
-1 - i h k n_0 & 1 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & \cdots \\
0 & \cdots & 0 & 1 & -2 & 1 \\
0 & \cdots & 0 & 1 & -1 - i h k n_0 \\
\end{pmatrix},
$$

$$
Q = \begin{pmatrix}
2 \hat{n}_0^2 & \hat{n}_0^2 & 0 & \cdots & 0 \\
\hat{n}_0^2 & 2 (\hat{n}_0^2 + \hat{n}_1^2) & \hat{n}_1^2 & 0 & \cdots \\
0 & \cdots & 0 & \hat{n}_{M_0-2}^2 & 2 (\hat{n}_{M_0-2}^2 + \hat{n}_{M_0-1}^2) & \hat{n}_{M_0-1}^2 \\
0 & \cdots & 0 & \hat{n}_{M_0-1}^2 & 2 \hat{n}_{M_0-1}^2 & \hat{n}_{M_0-1}^2 \\
\end{pmatrix},
$$

and $v = (\begin{pmatrix} -2 i k n_0 A_{\text{inc}} & 0 & \cdots & 0 \end{pmatrix}^T$ and $\hat{n}_j$ is the linear refractive index in the interval $(z_j, z_{j+1})$. In the following this scheme is called the standard FEM.

We will examine the order of accuracy of the standard FEM. For uniform medium with index $n$, the $j$th equation of (2.16) can be written as

$$
\delta^2 \tilde{E}_j + \frac{1}{6} k^2 \hat{n}^2 \left( \tilde{E}_{j-1} + 4 \tilde{E}_j + \tilde{E}_{j+1} \right) = 0,
$$

(2.17)

where

$$
\delta^2 \tilde{E}_j = \frac{1}{h^2} \left( \tilde{E}_{j+1} - 2 \tilde{E}_j + \tilde{E}_{j-1} \right).
$$

Using a Taylor expansion, it can be shown that

$$
\frac{d^2 E(z_j)}{dz^2} = \delta^2 \tilde{E}_j - \frac{h^2}{12} \frac{d^4 E(z_j)}{dz^4} + O(h^4),
$$

(2.18)

$$
\frac{d^2 E(z_j)}{dz^2} + k^2 \hat{n}^2 E(z_j) = \delta^2 \tilde{E}_j + \frac{1}{6} k^2 \hat{n}^2 \left( \tilde{E}_{j-1} + 4 \tilde{E}_j + \tilde{E}_{j+1} \right) + \text{Res},
$$

(2.19)

where $\text{Res} = -\frac{1}{12} k^2 \hat{n}^2 h^2 \frac{d^4 E(z_j)}{dz^4} + O(h^4)$. This shows that the standard scheme (2.17) for the interior point is only second order accurate. The accuracy of the standard discrete boundary conditions can be checked as follows. First we discretized the TIBC at $z = z_{\text{min}}$, see Equation (2.9), using a central difference

$$
\frac{1}{2h} \left( \tilde{E}_1 - \tilde{E}_{-1} \right) - i k n_0 E_0 = -2 i k n_0 A_{\text{inc}},
$$

(2.20)
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which is an \( O(h^2) \) approximation. Then this approximation is used to eliminate the term \( \hat{E}_{-1} \) in Equation (2.19) for \( j = 0 \) to recover the standard FEM at the left boundary (and similarly for the right boundary). Because both Equation (2.19) and (2.20) are of \( O(h^2) \), the standard FEM at the boundaries is also \( O(h^2) \).

Now we wish to improve the order of accuracy of the standard scheme (2.16). By applying the central difference to the first term of Res and adding it to (2.19) we obtain a fourth-order scheme

\[
\frac{d^2 E(z_j)}{dz^2} + k^2 n^2 E(z_j) = \left(1 - \frac{1}{12} k^2 n^2 h^2\right) \delta^2 \hat{E}_j + \frac{1}{6} k^2 n^2 \left( \hat{E}_{j-1} + 4\hat{E}_j + \hat{E}_{j+1} \right) + O(h^4)
\]

(2.21)

which leads to a linear system of equations (after modifying the boundary condition in a similar way)

\[
\left( \frac{1}{h} P_1 + \frac{1}{6} h k^2 Q \right) \hat{E} = v_1,
\]

(2.22)

where

\[
P_1 = \begin{pmatrix}
    -\alpha_0 (1 + i h k n_0) & \alpha_0 & 0 & \cdots & 0 \\
    \alpha_0 & -(\alpha_0 + \alpha_1) & \alpha_1 & \cdots & 0 \\
    0 & \cdots & 0 & \alpha_{M_0-2} - (\alpha_{M_0-2} + \alpha_{M_0-1}) & 0 \\
    0 & \cdots & 0 & \alpha_{M_0-1} & -\alpha_{M_0-1} (1 + i h k n_0)
\end{pmatrix}.
\]

with \( \alpha_j = 1 - \frac{1}{12} k^2 n^2 h^2 \) and \( v_1 = \left( -2i k n_0 A_{inc} \left( 1 + k^2 n_0^2 h^2 / 12 \right) 0 \cdots 0 \right)^T \).

Notice that the resulting matrix is still a tridiagonal and symmetric. This fourth-order scheme is called improved FEM.

Based on our numerical experiments we found that the numerical amplitude and phase error of the standard FEM is of \( O(h^2) \). When we apply the improved FEM to both a uniform structure and structure \((HL)^N(LH)^N\) with respectively \( N = 2 \) and \( N = 10 \), the numerical results show that the numerical error is indeed \( O(h^4) \), as expected.

2.3.2 Nonlinear scheme

In the previous section we have discussed a numerical method to solve the linear Helmholtz equation. Here we consider the complete NLH (2.7). The numerical method is basically the same as for the linear case but with an additional nonlinear term \( k^2 \chi^{(3)}(z) |E(z)|^2 E(z) \). The functional (2.11) has to be extended with

\[
\mathcal{F}_{NL}(E) = \frac{1}{4} \int_{z_{min}}^{z_{max}} k^2 \chi^{(3)}(z) |E|^4 dz.
\]

(2.23)
As in the previous section, we approximate the additional contribution $\mathcal{F}_{NL}$ as follows

$$
\mathcal{F}_{NL} (E) = \mathcal{F}_{NL} \left( \hat{E} \right)
= \frac{\hbar}{20} k^2 \sum_{j=0}^{M_0-1} \chi_j^{(3)} \tilde{\mathcal{F}}_{NL,j}
$$

(2.24)

where $\chi_j^{(3)}$ is the nonlinear coefficient in the interval $(z_j, z_{j+1})$ and

$$
\tilde{\mathcal{F}}_{NL,j} = \left| \hat{E}_j \right|^4 + \frac{1}{2} \left| \hat{E}_j \right|^2 \left( \hat{E}_j \hat{E}_{j+1} + \hat{E}_j \hat{E}_{j+1}^* \right) + \frac{1}{3} \left| \hat{E}_j \right|^2 \left| \hat{E}_{j+1} \right|^2 \\
+ \frac{1}{6} \left( \hat{E}_j^2 \hat{E}_{j+1}^* + 2 \left| \hat{E}_j \right|^2 \left| \hat{E}_{j+1} \right|^2 + \hat{E}_j \hat{E}_{j+1}^2 \right) \\
+ \frac{1}{2} \left| \hat{E}_{j+1} \right|^2 \left( \hat{E}_j \hat{E}_{j+1} + \hat{E}_j^* \hat{E}_{j+1}^* \right) + \left| \hat{E}_{j+1} \right|^4.
$$

(2.25)

The partial derivatives of $\tilde{\mathcal{F}}_{NL} \left( \hat{E} \right)$ are

$$
\frac{\partial \tilde{\mathcal{F}}_{NL}}{\partial \hat{E}_j} = \mu_{j-1} \left( \frac{2}{3} \hat{E}_{j-1} \hat{E}_j^* + \left| \hat{E}_{j-1} \right|^2 + 2 \left| \hat{E}_j \right|^2 \right) \hat{E}_{j-1}
+ \mu_{j-1} \left( \hat{E}_j \hat{E}_{j-1}^* + \frac{4}{3} \left| \hat{E}_{j-1} \right|^2 + 4 \left| \hat{E}_j \right|^2 \right) \hat{E}_j
+ \mu_j \left( \hat{E}_j \hat{E}_{j+1}^* + \frac{4}{3} \left| \hat{E}_{j+1} \right|^2 + 4 \left| \hat{E}_j \right|^2 \right) \hat{E}_j
+ \mu_j \left( \frac{2}{3} \hat{E}_{j+1} \hat{E}_j^* + \left| \hat{E}_{j+1} \right|^2 + 2 \left| \hat{E}_j \right|^2 \right) \hat{E}_{j+1}.
$$

(2.26)

where $\mu_j = \hbar k^2 \chi_j^{(3)}/20$. Adding the nonlinear term (2.26) to the linear scheme (2.22), we obtain:

$$
\left( \frac{1}{\hbar} P + \frac{1}{6} \hbar k^2 Q + R \left( \hat{E} \right) \right) \hat{E} = v,
$$

(2.27)

where $\nabla \tilde{\mathcal{F}}_{NL} \left( \hat{E} \right) = R \left( \hat{E} \right) \cdot \hat{E}$.

### 2.3.3 Nonlinear solver

We notice that our numerical scheme (2.27) is a system of nonlinear equations. The standard approach to solve a nonlinear system is a fixed-point iterative method which in our case is given by

$$
\left( \frac{1}{\hbar} P + \frac{1}{6} \hbar k^2 Q + R \left( \hat{E}^{(m)} \right) \right) \hat{E}^{(m+1)} = v,
$$

(2.28)

where $m = 1, 2, 3,...$ is the iteration step with $\hat{E}^{(1)}$ is taken from the solution of the linear problem. The iteration process is stopped when

$$
\left\| \hat{E}^{(m+1)} - \hat{E}^{(m)} \right\| < \varepsilon
$$

(2.29)
for small $\varepsilon$. For the calculation in this chapter we set the tolerance $\varepsilon = 10^{-6}$.

In general the fixed-point iterative algorithm leads to a convergent solution when there exist a unique solution. However, in the case of bistability where the solution is not unique, this iterative method may not converge. As an example, we implement the fixed-point iterative method (2.28) to calculate the transmissivity $|T|^2 = |\widehat{E}_M/A_{inc}|^2$ of a nonlinear structure $(HL)^6(D)^2(LH)^6$ with input intensity 1.004558 kW/m² (see Figure 2.16.(b) for other parameters). Remark that this example represents a situation around the jumping area from the low-output state into the high-output state.

We plot the calculated transmissivity using iterative scheme (2.28) for each iteration step in Figure 2.2 (dotted-line). It is seen that the fixed-point iterative method has a non-convergent solution. In order to take care of the divergence problem in the fixed-point iterative method, we replace the argument of $R$ in (2.28) with the weighted averaged of $\widehat{b}_E(m)$ with $\sigma$, satisfying $0 < \sigma \leq 1$, is a parameter used to control the weight of $\widehat{b}_E(m)$. Note that if we take $\sigma = 1$ then it leads to the standard fixed-point iterative method. Since $\widehat{E}^{(m-1/2)}$ is between $\widehat{E}^{(m)}$ and $\widehat{E}^{(m-1)}$, if the series $\widehat{E}^{(m)}$ is convergent then the series $\widehat{E}^{(m-1/2)}$ is also convergent; and furthermore both of those series have the same limit. Based on our numerical experiments, the iterative method (2.30) produces convergent solutions for sufficiently small values of $\sigma$. However the smaller $\sigma$ needs a larger number of iterations. As an example, we plot in Figure 2.2 the calculated transmissivity of a defect structure using the weighted-averaged fixed-point iterative method (2.30) as a function of the iteration number for $\sigma = 1/3$ and $1/10$, respectively. We see here that scheme (2.30) has a convergent solution with $|T|^2 = 0.6254$ after approximately 90 iterations for $\sigma = 1/3$ (dashed-line) and 200 iterations for $\sigma = 1/10$ (dashed-dotted-line). For $\sigma = 1/2$ the iterative scheme (2.30) does not leads to a convergent solution and the result is not shown.

We notice that the iterative method (2.30) is only converging to one solution, whereas in the case of bistability there are two stable solutions. Therefore we still need to find another solution besides the one obtained from (2.30). Considering the nature of bistability, i.e. the transmissivity depends not only on the input intensity alone but also on the history of it, Lu et al. [14] suggest to include the history of the input intensity in the calculation. Following this suggestion, we start by solving (2.30) for a very low incident amplitude $A_{inc}^{(0)}$ that is relatively far from the bistability using the linear solution as an initial guess $\widehat{E}^{(1)}$. Then we use the solution of this calculation as a starting point for $A_{inc}^{(1)} > A_{inc}^{(0)}$ and so on. In this algorithm, when the incident amplitude increases from $A_{inc}^{(0)}$, the solutions only correspond to the low-transmission level until the transmission jumps to the high-transmission level. On the contrary, if
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Figure 2.2: Transmissivity of a nonlinear defect structure $(HL)^6(D)^2(LH)^6$ with input intensity $1.004558 \text{kW/m}^2$ (for other parameters see Figure 2.16.(b)) as a function of the iteration number calculated by the standard fixed-point iterative method ($\sigma = 1$, dotted-line); by the weighted-averaged fixed-point iterative method with $\sigma = 1/3$ (dashed-line) and $\sigma = 1/10$ (dashed-dotted-line), respectively; and by combination of the weighted-averaged fixed-point iterative method ($\sigma = 1/3$) and the continuation method (full-line).

As we decrease the amplitude of the input wave from the high-transmission state then we can find another stable solution. Thus this algorithm, called the continuation method, can be used to assess the optical bistability. Another advantage of the continuation method is that the number of iterations is much smaller if we compare with the weighted-averaged iterative method alone (2.30), see Figure 2.2 (full-line).

As we will see in the following section, optical bistability in a finite grating can be tuned either by the frequency or the intensity of the input light. In the latter case, for a fixed frequency light propagation through the multilayer structures, optical bistability manifests itself by a non-unique dependence of the transmission on the intensity of the incident wave. Contrarily, when viewing the input intensity as a function of the transmission, the dependence was observed to be unique. Therefore, in order to improve the performance of finite element scheme (2.27), we use the transmitted wave as the input parameter instead of the incident wave. This approach is called a fixed output problem. As stated in (2.13), the transmitted wave in the homogeneous medium beyond the defect structure is of the form

$$E(z) = A_{tr} \exp(-ikn_0(z - z_{max})), \quad z \geq z_{max}. \quad (2.31)$$

Without loss of generality, we assume that the transmitted wave amplitude $A_{tr}$ is a
real constant. Consequently the value of the incident wave amplitude $A_{\text{inc}}$ now can be a complex number. In this approach the nonlinear system (2.22) is reformulated such that $A_{\text{inc}}$ is included as an unknown variable and $A_{\text{tr}}$ is the input parameter:

$$
\begin{pmatrix}
2i/ k \left( 1 + k^2 n_0^2 \hbar^2 / 12 \right) \\
0 \\
\vdots \\
0 \\
(0)
\end{pmatrix}
\begin{pmatrix}
\frac{1}{h} P_1 + \frac{1}{2i} k^2 Q + R \left( E \right) \\
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
A_{\text{inc}} \\
\hat{E}_0 \\
\hat{E}_1 \\
\vdots \\
\hat{E}_{M_0-1} \\
\hat{E}_{M_0}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\vdots \\
0 \\
A_{\text{tr}}
\end{pmatrix}.
$$

(2.32)

We solve the nonlinear system (2.32) using a standard fixed-point iterative method. Since the scheme (2.32) produces a unique relation between the transmitted wave and the incident wave, we can expect that the convergence of this approach is much faster than that of the fixed input problem. As an example, the scheme (2.32) only needs four iterations to solve a problem related to Figure 2.2, while scheme (2.30) using $\sigma = 1/3$ combined with a continuation method requires 40 iterations! Furthermore, as shown in next section, the fixed output approach can find all stable and unstable solutions related to a specific incident, whereas the fixed input method does not give access to unstable solutions. Hence, for the calculation of the optical bistability controlled by the the input intensity, we will only implement the fixed output scheme (2.32).

### 2.4 Optical bistability in 1D PBG structure

In this section we apply the numerical scheme that was derived in the previous section to study the transmission properties of quarter-wavelength stacks as shown in Figure 2.1. For both linear and nonlinear numerical calculations we take $n_L = 1.25$ and $n_H = 2.5$. Unless it is mentioned otherwise, the refractive index of the input and output regions $z < z_{\text{min}}$ and $z > z_{\text{max}}$ is assumed to be $n_0 = 1$ and that of the defect layer is $n_d = n_H$. The computational window is divided into $M_0$ equidistant elements with grid size $h = \lambda_0/400$. All results given have been found to have converged with respect to a refinement of the discretization and with respect to the iteration number.

Before proceeding with the nonlinear case, we first discuss the linear structure by setting $\chi^{(3)} = 0$. For a linear structure $(HL)^4(D)(LH)^4$ with total computational window $3.5 \lambda_0$ (the length of each uniform layer in the left and the right parts of the grating structure is $\lambda_0/2$), applying the improved FE scheme in place of the original one allows us to reduce the number of elements from about 5600 that are necessary to compute equally accurate results with scheme (2.16) to a number of 1400 for scheme (2.22). Within this accuracy, if we compare the results of the analytical transfer matrix method, the improved FEM gave no observable differences on the scale of figures presented in this section. For the nonlinear structure, however, we do
not have an exact analytical solution. Therefore we compared our numerical results of bistability controlled by the input intensity with those of the nonlinear transfer matrix method [7], using the full nonlinear interface conditions [1]. We remark that the nonlinear transfer matrix method is valid only for the case of weak nonlinearity. We observed a very good agreement between those two methods for the case of a defect structure but not for a perfect structure. This can be understood from the fact that the maximum change of refractive index due to nonlinearity in an ideal structure is much bigger than that of a defect structure. Another remark is that the FEM scheme can be used to assess the bistability tuned by input frequency. However, this calculation is very difficult to be done by the nonlinear transfer matrix method.

Figure 2.3: (a) Transmission spectra of structure \((HL)^N(D)(LH)^N\) with \(N = 4\) and \(N = 8\), respectively. The first band gaps of structure with \(N = 4\) and \(N = 8\) are respectively \(\omega \in (0.75915, 1.24085) \times 2\pi c/\lambda_0\) and \(\omega \in (0.77633, 1.22367) \times 2\pi c/\lambda_0\). (b) The normalized amplitude of the electric field inside a perfect structure with \(N = 4\) at \(\omega = 1.24085 \times 2\pi c/\lambda_0\). (c) Same as for (b) but with \(N = 8\) at \(\omega = 1.22367 \times 2\pi c/\lambda_0\).

2.4.1 Transmission properties of perfect linear PBG structure

As mentioned in section 2.1, a perfect (infinite) PBG structure has an essential property, i.e. the existence of forbidden bands prohibiting a certain range of frequencies of light waves to propagate through them. In other words, the light waves with frequencies inside the band gap are completely reflected by the structure. However, in a finite
periodic structure, the reflection will not be complete in general. Therefore we practically use the term band gap (of a finite structure) for the smallest frequency interval containing the band gap of the infinite structure that is bordered by two resonance frequencies (which are called band-edge modes/resonances). To get a feeling of this definition, we show in Figure 2.3.(a) the transmission spectra around the first band gap of structure \((HL)^N(D)(LH)^N\) with \(N = 4\) and \(N = 8\), respectively. As shown in this picture, the increasing number of periods reduces the width of the band gap. If \(N\) is further increased, the band gap interval is closer to the actual band gap, i.e. the interval for an infinite structure, which in this case is between \(\omega = 0.78365 \times 2\pi c/\lambda_0\) and \(\omega = 1.21635 \times 2\pi c/\lambda_0\). By analogy to the solid state electronics case, the frequency which lies below the band gap is called the valence band (VB) and the upper one is the conduction band (CB). We notice that by adding more layers, the number of resonance frequencies where the transmission equals one is also increased and specifically the resonance at the band-edge becomes sharper.

2.4.2 Transmission properties of linear PBG structure with a defect

When a defect layer is introduced in a linear PBG structure, a very narrow resonance that is isolated in the band gap occurs. The frequency of such resonance is called a defect mode frequency (see e.g. [3] and [22]). This is in contrast with the case of strictly periodic structure where all resonances are concentrated at the border or outside the band gap. In the following paragraphs we discuss the dependence of the defect modes on the thickness, the position and the refractive index of the defect layer and the number of layer periods.

To study the defect mode, we first consider an ideal PBG structure with 17 alternating layers \((HL)^4(D)^M(LH)^4\). Then by disturbing the width of the center layer, which has a high refractive index \(n_H\), we obtain a symmetric defect structure \((HL)^4(D)^M(LH)^4\). In Figure 2.4 we show the position of transmission maxima where the transmission coefficient is unity as a function of the defect layer width. The perfect structure, i.e. when the thickness of the center layer is \(\lambda_0/4n_H\) (see dashed line in Figure 2.4) has a band gap which is centered at the frequency \(2\pi c/\lambda_0\). The shaded-region indicates the width of the first band gap. The band-edge resonances in this case are respectively at \(\omega = 0.75915 \times 2\pi c/\lambda_0\) and \(\omega = 1.24085 \times 2\pi c/\lambda_0\). When reducing the defect width, the valence band edge moves into the band gap to become a defect mode. Since the defect mode evolves from the valence band with decreasing the defect size, it can be thought of as an acceptor mode (see e.g. Ref. [26]). We note that the acceptor mode frequency increases as the defect size is decreased. On the other hand, if the width of the center layer of our perfect structure is increased the conduction band
Figure 2.4: The (frequency) position of transmission maxima as a function of the defect layer width for structure \((HL)^4(D)^M(LH)^4\) with \(n_d = n_H\). The media in the input and output regions are assumed to be air \((n_0 = 1)\). The shaded-area indicates the first band gap of the perfect structure, i.e. when the defect layer is \(\lambda_0/4n_d\) (see the dashed-line). Observe the appearance of acceptor/donor modes caused by changing the defect size. It is noticed that when two transmission maxima outside the band gap meet, they show an anticrossing behavior (see e.g. the circled areas).

Figure 2.5: The same as Figure 2.4 except that the first and last semi-infinite layers are now assumed to have refractive index \(n_0 = n_L\) instead of \(n_0 = 1\). Similar to that presented in Figure 2.4, acceptor/donor modes are created as we change the thickness of the defect layer. With disturbing the defect layer, two transmission maxima can merge and split but the frequency of one of these two maxima remain the same.
edge moves into the band gap region. This type of defect mode is called a donor mode. When the width of the defect layer is further increased more than one defect mode can be obtained. Stanley et al. [19] have noticed that the moving behavior of the transmission maxima of the defect structure is identical to the case of solid-states electronic, except for the anticrossing behavior outside the band gap. However, when we assume that the front and the back media have a refractive index $n_L$ instead of air, the anticrossing behavior cannot be observed anymore (see the circled areas in Figure 2.5). In this case, as we change the defect layer, two transmission maxima outside the band gap merge and then split again with the frequency of one of these maxima remaining constant.

It is well known that the defect mode can enhance the field intensity inside the structure. By assuming that the incident wave amplitude equals to one, we show in Figure 2.6 the enhancement of the field amplitude as a function of the size of the defect layer for donor mode 1, donor mode 2 and donor mode 3, respectively. The field amplitude enhancement here is defined as the maximum field amplitude $\max(E)$ inside the structure. It is shown in Figure 2.6 that for each donor mode the largest enhancement occurs when the width of the defect layer equals to $\alpha (\lambda_0/2n_H)$ for integer $\alpha$. It can be checked in Figure 2.4 that those donor modes have the same frequency $\omega = 2\pi c/\lambda_0$. The maximum enhancement of the field amplitude in this case is sixteen times of the incident wave. Although the enhancement factors of structures with defect thicknesses $\alpha (\lambda_0/2n_H)$ are the same, the field amplitudes...
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Figure 2.7: The field amplitude $|E|$ inside the defect structure $(HL)^4(D)^M(LH)^4$ with $n_d = n_H$ for (a) $L_d = \lambda_0/2n_H$; (b) $L_d = \lambda_0/n_H$; (c) $L_d = 3\lambda_0/2n_H$ and (d) $L_d = 2\lambda_0/n_H$ at frequency $\omega = 2\pi c/\lambda_0$. Observe that the maximum of the amplitude is 16 for all cases. The longer the defect layer is the more amplitude maxima can be observed inside the defect region.

inside the structures are different for different $\alpha$. By increasing the value of $\alpha$ by one an additional field amplitude peak is observed in the structure, see Figure 2.7. Moreover, the change of $\alpha$ will also change the spectral width of the defect mode. It is clearly seen in Figure 2.8 that the full width at half maximum (FWHM) of the defect mode decreases with increasing $\alpha$.

Now we study the influence of the defect position in the structure on the defect mode. We are still considering a structure of 17 alternating layers including 9 $H$ layers and 8 $L$ layers. The defect layer is introduced by changing the size of one of the layer $H$ to be $\lambda_0/2n_H$. The effect of defect position is investigated by moving the defect layer from the left to the right of the structure. It is found that the changing of the defect position disturbs the positions of the transmission maxima outside the band gap. However the position of the defect mode remains the same, i.e. at frequency $2\pi c/\lambda_0$.

Furthermore, the transmission coefficients of those transmission maxima including the defect mode can be less than one when the defect layer is not located at the center of the structure. Because we are more interested in the defect mode, we plot in Figure 2.9 the transmission coefficient of the defect mode as a function of the position of the defect layer. The incident wave is fully transmitted by the defect structure only when the defect layer is placed in the center of the structure. Otherwise the incident light is partly reflected. Wang et al. [24] explained this phenomena by considering the whole defect structure as two smaller structures linked together, i.e. one is structure
Chapter 2

Frequency \(2\pi c/\lambda_0\)

Transmission coefficient

<table>
<thead>
<tr>
<th>Frequency</th>
<th>0.997</th>
<th>0.998</th>
<th>0.999</th>
<th>1</th>
<th>1.001</th>
<th>1.002</th>
<th>1.003</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.1</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
</tr>
<tr>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7</td>
<td>0.8</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 2.8: The transmission spectrum around the defect mode for different defect layer widths. The spectral width of the defect mode decreases with the increasing defect \(L_d\).

with a defect in the middle and the other is a perfect structure. Incident light with frequency of the defect mode can pass through the defect part, but is partly reflected by the perfect part because its frequency is in its band gap. Therefore, the structure with a central defect layer has the highest transmission coefficient.

Next we investigate the dependence of the defect mode on the refractive index of the defect layer. The defect structure considered here has the form \((HL)^4(D)^2(LH)^4\). The index of the defect layer \(n_d\) is varied from 2 to 4. According to the results of our calculations, an acceptor mode appears in the band gap if the defect thickness is less than \(\lambda_0/4n_d\). On the contrary, some donor modes can be obtained if the size of the defect layer is greater than \(\lambda_0/4n_d\). Similar to the previous case the enhancement factor of the field amplitude is sixteen and is obtained when the width of the defect layer is a multiple of \(\lambda_0/2n_d\) (i.e. when \(M\) is an even number) at defect mode frequency \(2\pi c/\lambda_0\). Furthermore the FWHM of the defect mode of a structure \((HL)^4(D)^2(LH)^4\) becomes smaller as we increase the refractive index of the defect layer, see Figure 2.10.

Finally we investigate the effect of the number of layer periods by considering the symmetrical structure \((HL)^N(D)^2(LH)^N\) for \(N = 4, 5, 6, 7\). The index and the width of the defect layer are chosen to be \(n_H\) and \(2(\lambda_0/4n_H)\), respectively. Using this structure, a defect mode in the center of the band gap of the corresponding perfect structure can be found. Figure 2.11 shows the field amplitude inside the structure for different \(N\). The maximum field intensity is changed as the number of periods changes. A larger \(N\) produces a larger field amplitude enhancement. More precisely
Figure 2.9: The transmission coefficient at $\omega = 2\pi c/\lambda_0$ of structure $(HL)^N (D)^2 (LH)^{8-N}$ for $N = 0, 1, \ldots, 8$. The maximum transmission occurs when the defect layer is placed in the middle of the structure.

Figure 2.10: The transmission spectrum around the defect mode of a structure $(HL)^4 (D)^2 (LH)^4$ for different refractive indices of the defect layer. The spectral width of the defect mode decreases when the refractive index of the defect layer is increased.
the enhancement factors of structures with \( N = 4, 5, 6, 7 \) are respectively 16, 32, 64 and 128 which are exactly \( 2^N \). In addition to the increasing field enhancement of the defect mode with respect to the increasing \( N \), its spectral width also decreases, see Figure 2.12.

It is well known that the enhancement factor and the decrease of the spectral width are important properties for optical bistability. Based on the previous discussion we conclude that to get a high field enhancement a defect structure \((HL)^N (D)^M (LH)^N\) is to be designed such that the defect layer is positioned in the middle of the structure, i.e \( N_1 = N_2 \) with \( M \) being an even integer number. Then the defect mode is located in the center of the band gap. A higher value of \( M \) leads to a smaller FWHM. Increasing the number of grating periods \( N \) yields simultaneously a narrower resonance and a field enhancement that grows exponentially with \( N \).

### 2.4.3 Bistable switching by frequency tuning

If a Kerr nonlinearity is introduced in the structure, it causes a change of local refractive index. According to Ref. [10] the induced refractive index change is proportional to the intensity of the optical field and can be written as

\[
\Delta n = \pi\varepsilon_0 |E|^2 \tag{2.33}
\]
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Figure 2.12: The transmission spectrum around the defect mode of the structures $(HL)^N (D)^2 (LH)^N$ for $N = 4, 5, 6, 7$. The spectral width of the defect mode decreases with growing number of periods $N$.

where the nonlinear refractive index coefficient $\overline{n}_2$ of the medium is defined by

$$\overline{n}_2 = \frac{1}{2n} \chi^{(3)}.$$  \hspace{1cm} (2.34)

For the numerical calculations of bistability in this chapter, we assume a Kerr nonlinearity with $\chi^{(3)} = 2 \times 10^{-12}$ m$^2$V$^{-2}$ is present in all high index layers $H$ as well as in the defect layer (when considering a defect structure). Due to a change of the effective refractive index, the transmission spectrum will also change accordingly. In general, a positive (negative) nonlinearity has a tendency to shift nonlinearly the entire transmission spectrum to the left (right). Specifically when the resonance is sharp enough, bistability or multistability can be achieved (see, e.g. Agranovich et al. [2]). For a strictly periodic structure, bistability usually occurs around the resonance peak at the border of the linear band gap: a positive (negative) nonlinearity effects bistability in the left (right) flanks of the resonance frequencies. For example we show in Figure 2.13 the transmission spectra of the nonlinear structures $(HL)^N (D) (LH)^N$ for $N = 4$ and $N = 8$ respectively in the vicinity of the resonance peak at the top of the first linear band gap. Each spectrum is calculated using a fixed input intensity $I_{inc}$. For a certain threshold of $I_{inc}$, the spectrum exhibits a bistability phenomenon: by increasing the frequency of a tunable source, the transmission jumps into high-transmission state $1'$ after it passes point 1. Similarly, the low-transmission state $2'$ can be reached after passing state 2 when moving in the opposite direction, see Figure 2.13.(a). For $N = 4$ bistability can be obtained when we set $I_{inc} = 7.5 \times 10^4$ kW/m$^2$ but not when $I_{inc} = 4 \times 10^4$ kW/m$^2$. When the number of layers is increased such that $N = 8$, the transmission spectrum already exhibits bistability for $I_{inc} = 5 \times 10^3$...
For a nonlinear defect structure the bistability can occur in the vicinity of the resonance peak at the border of the linear band gap or around the defect mode frequency where the resonance is relatively sharp. As we notice before the electric field intensity inside a symmetric defect structure at the defect mode frequency is very high so that we can use it to enhance the nonlinear effect. Therefore bistable behavior can be expected around this frequency with a lower threshold compared with a periodic structure of the same length. And indeed this is confirmed by our numerical calculations. Figure 2.14 shows the transmission spectra of the nonlinear defect structure \((HL)^4(D)(LH)^4\) for different \(I_{inc}\)'s. The bistability can already be observed for \(I_{inc} = 100\) kW/m\(^2\).

2.4.4 Bistability controlled by input intensity

The discussion in the previous subsection suggests the possibility for the use of a finite periodic structure as a device in which bistable switching is controlled by frequency tuning while input electric field is maintained at fixed intensity. Furthermore, we also observe that for a certain frequency, e.g. \(\omega = 0.9988 \times 2\pi c/\lambda_0\) in Figure 2.14, the transmissivity is found uniquely for relatively low \(I_{inc}\), then it is multi-valued.
Figure 2.14: Transmission spectra of nonlinear structure \((HL)^4 (D)^2 (LH)^4\) for different input intensity. The threshold for bistability of a defect structure is much lower than that of a perfect structure with comparable length.

for larger \(I_{inc}\). However, by increasing the input intensity, the transmissivity can be unique again. This fact also shows the possibility of optical bistability controlled by input light intensity at fixed frequency.

A basic issue of the optical bistability is to realize it with threshold as low as possible. According to the previous discussion, a symmetric defect structure has a much lower threshold of bistability compared to the perfect structure of the same length. Therefore, we restrict our study of bistability controlled by the intensity of the input light only in a symmetric defect structure.

In Figure 2.15 we show the input-output characteristics of structure \((HL)^4 (D)^2 (LH)^4\) for some frequencies below the defect mode \(\omega = 2\pi c/\lambda_0\). It is found that for \(\omega = 0.995 \times 2\pi c/\lambda_0\) the structure shows an \(S\)-shape response. When the incident intensity \(I_{inc}\) increases slowly from zero, the transmitted intensity \(I_{tr}\) first increases slowly. If the input reaches the upswitching threshold value (about 6228.3 kW/m\(^2\)), \(I_{tr}\) jumps into a higher value (from state 1 to 1', see Figure 2.15.(a)). Then \(I_{tr}\) increases slowly again as we increase the value of \(I_{inc}\). On the other hand, when \(I_{inc}\) is decreased from the value that is greater than the threshold value, \(I_{tr}\) decreases slowly from the high value. When \(I_{inc}\) reaches the threshold value (state 1'), \(I_{tr}\) does not jump back to lower value (state 1), but it remains to decrease slowly until it reaches state 2, at which it jumps to state 2'. Then \(I_{tr}\) continues to decrease with decreasing \(I_{inc}\). Thus, the nonlinear defect structure can implement an optical bistability. It should be noticed that the line between the low-output state and high-output state, i.e. the line which connects the state 1 and state 2, corresponds to the unstable solutions.

While the upswitching threshold value for \(\omega = 0.995 \times 2\pi c/\lambda_0\) is very large, this value can be reduced by tuning the frequency of the input light closer to the defect mode. For example, the thresholds for \(\omega = 0.997 \times 2\pi c/\lambda_0\) and \(\omega = 0.999 \times 2\pi c/\lambda_0\) are 1392.8 kW/m\(^2\) and 81.2 kW/m\(^2\) respectively. However, the bistable behavior can
not be obtained anymore when the input field has frequency that is very close to the resonance frequency, e.g. in the case of $\omega = 0.9995 \times 2\pi c/\lambda_0$, see Figure 2.15,(b).

We remark that the change of refractive index due to the Kerr nonlinearity that corresponds to the incident intensity $6228$ kW/m$^2$ is $0.0412$, which is relatively large. When the intensity threshold is reduced to $81.2$ kW/m$^2$, the corresponding refractive index change is $0.0063$.

Now we investigate the effect of the defect thickness to the threshold of the bistability. We show in Figure 2.16,(a) the bistability curve of structure $(HL)^4(D)^M(LH)^4$ for $M = 2, 4, 6$. The optical bistability thresholds are $\sim 62.28$ kW/m$^2$ for $M = 4$ and $\sim 28.29$ kW/m$^2$ for $M = 6$. It was noticed in the previous section that increasing the defect size does not influence the field enhancement factor but it reduces the FWHM of the defect mode. Therefore, when we increase the value of $M$, we should select $\omega$ to be closer to the defect frequency, see Figure 2.16,(a). We conclude that the narrower the width of the defect mode, the lower the threshold of the bistability will be achieved. This qualitative behavior agrees with the result of the FDTD analysis done by Lixue et al. [13].

Based on the previous analysis, it can be stated that for a fixed Kerr constant, the bistability threshold is reduced when the width of the defect mode is smaller or when the enhancement factor is larger. Since an increasing number of grating periods

Figure 2.15: The input-output characteristics of a structure $(HL)^4(D)^2(LH)^4$ with $n_d = n_H$ for different frequencies where the Kerr nonlinearity is introduced in all high index layers. The bistability threshold decreases when the input light frequency is closer to the defect mode. However, when the frequency is too close to the defect mode (the case of $\omega = 0.9995 \times 2\pi c/\lambda_0$) the bistability cannot be obtained anymore.
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![Graph](image_url)

Figure 2.16: The input-output characteristics of structures $(HL)^N (D)^M (LH)^N$ with $n_d = n_H$ for different $M$ and $N$ where the Kerr nonlinearity is introduced in all high index layers: (a) $N = 4$ and $M = 2, 4, 6$; (b) $M = 2$ and $N = 5, 6$. In (b) the axis on the left hand side is meant for the structure $N = 5$ and $M = 2$, the transmission for structure $N = 6$ and $M = 2$ is to be found on the right hand axis.

produces a smaller FWHM and simultaneously enlarges the enhancement factor, we can expect that a higher number of periods will produce optical bistability with a lower threshold. And indeed, a very low threshold is already obtained when we use a defect structure $(HL)^N (D)^2 (LH)^N$ for $N = 5$ or $N = 6$. The bistability thresholds in these cases are about $7.55$ kW/m$^2$ for $N = 5$ and about $0.96$ kW/m$^2$ for $N = 6$ (see Figure 2.16(b)). The change of refractive index which corresponds to the latter case is only $\sim 7.6 \times 10^{-4}$.

2.5 Concluding remarks

We have presented a simple numerical scheme based on the variational method to study the optical response of a finite one-dimensional nonlinear grating. Restricting first to the linear case, we improve the standard FEM to get a fourth order accurate scheme maintaining a symmetric-tridiagonal structure of the finite element matrix. For the full nonlinear equation, we implement the improved FEM for the linear part and a standard FEM for the nonlinear part. When using the amplitude of the incident wave as the input parameter, we solve the resulting nonlinear system of equations by implementing a weighted-averaged fixed-point iterative method combined with a continuation method. This approach can capture the bistability phenomenon in nonlinear grating structure as a function of both the frequency or the intensity of the
input light. However, this method can only find the two stable solutions but not the unstable solutions. In case of multistability, it will be very difficult to find more than two stable solutions. In addition, the convergence of the iteration procedure can be very slow in the region of upswitching from low-output level to high-output level. For the case of bistability controlled by the intensity of the input light, the performance of our scheme is improved by using the transmitted wave as the numerical input parameter instead of the incident wave. This approach leads to a unique solution and only needs a standard fixed-point iterative method. Therefore the convergence of the fixed-output approach is much faster that that of the fixed-input method.

When considering a linear PBG structure, we also find that a defect layer can create defect modes. It is found that the shape of a defect mode depends on the thickness, the position and the refractive index of the defect layer as well as on the number of the grating periods. When the defect thickness is less (greater) than $\lambda_0/4n_d$ then acceptor (donor) modes can be observed in the band gap of the perfect structure. It is found that an optimal field enhancement is obtained when the defect layer is placed in the middle of the structure with the defect thickness being a multiple of $\lambda_0/2n_d$. Increasing the defect layer thickness yields a smaller spectral width of the defect mode. A larger enhancement factor and simultaneously a narrower FWHM can be achieved by increasing the number of layer periods.

If a Kerr medium is present either in a perfect structure or in a defect structure, it can cause a bistability phenomenon. The bistability of an ideal grating structure can be obtained by tuning the frequency to a value close to the bottom or top linear band-edge while that of a defect structure can be produced using a frequency near the defect mode or near the linear band-edge. We found that the threshold needed for a defect structure which has good optical features (large field enhancement and narrow resonance) is much lower than that for a strictly periodic structure of the same length. The threshold value can be reduced by increasing the number of layer periods.

We remark that the presented FE scheme implements the 1D NLH equation and exact TIBC without introducing any approximation except the finite element discretization. This is different from the nonlinear transfer matrix formalisms mentioned in section 2.1 that are based on the SVEA and other approximations. Therefore, our method can be used to study the validity of those approximative methods.

While our analysis focuses on layer stacks with step-like refractive indices, the presented method can be applied easily to more general structures, i.e. not necessarily periodic nonlinear gradient or piecewise constant permittivity profiles.
References


