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From recursion operators to Hamiltonian structures. The factorization method

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FROM RECURSION OPERATORS TO HAMILTONIAN STRUCTURES. THE FACTORIZATION METHOD.

P. KERSTEN AND I. KRASIL′SHCHIK

Abstract. We describe a simple algorithmic method of constructing Hamiltonian structures for nonlinear PDE. Our approach is based on the geometrical theory of nonlinear differential equations and is in a sense inverse to the well-known Magri scheme. As an illustrative example, we take the KdV equation and the Boussinesq equation. Further applications, including construction of previously unknown Hamiltonian structures, are in preparation.

Contents

Introduction

The most popular method of establishing integrability of a partial differential equation is construction (or, rather, revealing) of a bi-Hamiltonian structure, [2, 3, 9, 10], we stick to the common terminology). The scheme itself seems to be quite clear and well-tested; nevertheless one may certainly meet (and meets!) serious technical difficulties in particular applications. These difficulties become even more

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severe when dealing with supersymmetric systems (see examples of the latter in [6, 7]) and construction of Hamiltonian operators for some of these systems still remains a challenge. Computations also become quite complicated if Hamiltonian operators happen to be nonlocal.

An alternative approach to Hamiltonian structures in the field theory (i.e, in partial differential equations) was first formulated in [11] (see also [1]), but was not widely accepted — probably, because it uses intricate techniques of differential geometry [4] and homological algebra [12]. It is also worth to note that the criteria of hamiltonianity given in [11] are rather complicated for *finding* Hamiltonian structures, but quite convenient for *checking* whether a particular operator is a Hamiltonian one.

The method described in this paper originates from a simple observation. The bi-Hamiltonian property of equations is used for the following purposes:

(a) to establish existence of infinite series of conservation laws,
(b) to prove that they are in involution[1];
(c) to construct recursion operators for symmetries in the form $R = H_2 \circ H_1^{-1}$ whatever the last factor means (here H_1 and H_2 are the first and second Hamiltonian structures respectively).

Since recursion operators are one of the outcomes of the scheme, why not to reverse the procedure and to start with constructing R? One of the reasons for this is the fact that to find R one needs only to solve rather simple *linear* equations [7](while for Hamiltonian structures the equations are *nonlinear*). If we manage to do it, let us try to decompose the result in two factors and check whether the first one is a Hamiltonian structure. Moreover, as the reader will see, to find objects of the form H_1^{-1} is much simpler than H_1 itself (they are also described by linear equations)! So, let us find H_1^{-1} and "divide" R by H_1^{-1}. Then, knowing H_2, we "divide" it by R and find H_1. Surprisingly, these naive scheme works, and works efficiently.

In Section 1 we introduce the main concepts used in the paper. Section 2 describes our method of construction of Hamiltonian structures. Two examples of illustrative nature are considered in Section 3. The first one is the classical KdV equation, the second is the Boussinesq equation. We show that our method brings us to the known Hamiltonian structures for these equations. In particular, it was nice to obtain the tri-Hamiltonian structure of the Boussinesq equation in a straightforward way. Finally, Section 4 is devoted to the short discussion of the obtained results.

Further applications of the method exposed here, including construction of previously unknown Hamiltonian structures, are in preparation and are to appear elsewhere.

1. Preliminaries

Adapted to our aims, we expose briefly the basics of geometrical theory of nonlinear PDE and deformational approach to recursion operators. A detailed exposition can be found in the monographs [4] and [7].

[1](a)+(b)=complete integrability.

1.1. **Equations.** Though everything exposed below remains valid in the general case, we restrict ourselves to the case of evolutionary equations (or systems) in one space variable

$$u_t = f(t, x, u, u_1, \dots, u_k), \tag{1}$$

where $u_s = \partial^s u/\partial x^s$ and both u and f may be understood as vector-functions $u^1, \dots, u^m$, $f^1, \dots, f^m$ in the case of systems.

We shall work in the "on-shell" setting. Effectively, this means that using equation (1) and all its differential consequences we express all partial derivatives containing t in terms of u_s and thus arrive at the infinite-dimensional space $\mathcal{E}^\infty$ (the *infinite prolongation* of (1)) with the coordinates $t, x, u_0, \dots, u_s, \dots$, where $u_0 = u$. Though all these spaces are equivalent to each other, information about the initial equation still remains encoded in the two vector fields on $\mathcal{E}^\infty$

$$D_x = \frac{\partial}{\partial x} + \sum_{s\geq 0} u_{s+1}\frac{\partial}{\partial u_s}, \tag{2}$$

$$D_t = \frac{\partial}{\partial t} + \sum_{s\geq 0} D_x^s(f)\frac{\partial}{\partial u_s} \tag{3}$$

called the *total derivatives.* These vector fields commute with each other while the triple $(\mathcal{E}^\infty, D_x, D_t)$ is an isomorphic image of (1). A differential operator on $\mathcal{E}^\infty$ will be called a *$\mathcal{C}$-differential operator* (or *total differential operator*) if it can be expressed in terms of total derivatives.

An alternative way to describe the equation structure on $\mathcal{E}^\infty$ is to consider the differential 1-forms

$$\omega_s = du_s - u_{s+1}\,dx - D_x^s(f)\,dt, \quad s = 0, 1, \dots, \tag{4}$$

called the *Cartan forms*[2]. A vector field X on $\mathcal{E}^\infty$ annihilates all Cartan form, i.e., $X \lrcorner\, \omega_s = 0$ for all s if and only if it is a linear combination of total derivatives: $X = gD_x + hD_t$. Here $\lrcorner$ denotes the *inner product*, *contraction* of vector fields with differential forms.

1.2. **Symmetries.** We say that a vector field X on $\mathcal{E}^\infty$ is *vertical* if it does not contain $\partial/\partial x$ and $\partial/\partial t$ components: $X = \sum_s X_s \partial/\partial u_s$. Such a field is a *symmetry* of equation (1) if it commutes with total derivative operators:

$$[X, D_x] = 0, \qquad [X, D_t] = 0. \tag{5}$$

From the first equation it follows that X is of the form

$$X = \sum_{s=0}^{\infty} D_x^s(\varphi)\frac{\partial}{\partial u_s}, \tag{6}$$

where φ is an arbitrary function on $\mathcal{E}^\infty$ (here and below, by a function on $\mathcal{E}^\infty$ we understand a smooth function depending on finite number of variables x, t, u_i). In the case of systems, $\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector-functions and the corresponding formula is

$$X = \sum_{j=1}^{m}\sum_{s=0}^{\infty} D_x^s(\varphi^j)\frac{\partial}{\partial u_s^j}. \tag{7}$$

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[2]We shall also call ω a *Cartan form* if it is a linear combination of ω_s.

Vector fields of the form (6) or (7) will be called *evolutionary* and φ is called the *generating function* of such a field. The evolutionary vector field with the generating function φ will be denoted by ∂_φ.

The second condition in (5) means that a vector field ∂_φ is a symmetry if and only if its generating function satisfies the equation

$$D_t(\varphi) = \sum_{s=0}^{k} \frac{\partial f}{\partial u_s} D_x^s(\varphi), \tag{8}$$

where k is the order of equation (1). If a function φ satisfies (8), we shall say that it is a *symmetry* of (1) meaning that ∂_φ is its symmetry. The operator

$$\ell_{\mathcal{E}} = D_t - \sum_{s=0}^{k} \frac{\partial f}{\partial u_s} D_x^s \tag{9}$$

is called the (*universal*) *linearization operator* for equation (1). Thus the symmetry condition (8) may be rewritten as

$$\ell_{\mathcal{E}}(\varphi) = 0. \tag{10}$$

Note that the $\mathbb{R}$-vector space $\mathrm{sym}(\mathcal{E})$ of symmetries is endowed by a Lie algebra structure defined by

$$\{\varphi, \psi\} = \partial_\varphi(\psi) - \partial_\psi(\varphi) \tag{11}$$

and $[\partial_\varphi, \partial_\psi] = \partial_{\{\varphi,\psi\}}$.

We also associate the linearization operator to any function $\psi = \psi(t, x, u, \dots, u_r)$ on $\mathcal{E}^\infty$ by setting

$$\ell_\psi = \sum_{s=0}^{r} \frac{\partial \psi}{\partial u_s} D_x^s. \tag{12}$$

If ψ is a vector-valued function, then ℓ_ψ is a matrix operator of the form

$$\ell_\psi = \left\| \sum_{s=0}^{r} \frac{\partial \psi^\alpha}{\partial u_s^\beta} D_x^s \right\|.$$

Directly from definitions we have that linearizations are $\mathcal{C}$-differential operators.

Note also that the identity

$$\ell_\psi(\varphi) = \partial_\varphi(\psi)$$

holds for all functions φ and ψ.

Remark. There is a well-defined action of evolutionary vector fields on vector-functions. This action is component-wise and is given by

$$\partial_\varphi(\varphi^1, \dots, \varphi^s) = (\partial_\varphi(\varphi^1), \dots, \partial_\varphi(\varphi^s)).$$

1.3. **Adjoint operators and variational derivative.** For any matrix $\mathcal{C}$-differential operator

$$\Delta = \left\| \sum_{i,j} (a_{ij})_\alpha^\beta D_x^i D_t^j \right\|$$

we define its *adjoint* by

$$\Delta^* = \left\| \sum_{i,j} (-1)^{i+j} D_t^j D_x^i \circ (a_{ij})_\beta^\alpha \right\|. \tag{13}$$

If Δ is an operator in D_x only, the following simplified version of *Green's formula* holds

$$\Delta(\varphi)\cdot\psi-\varphi\cdot\Delta^*(\psi)=D_x(\eta), \tag{14}$$

i.e., for any φ and ψ there exists an η such that (14) holds.

The *Euler operator*, or *variational derivative*, of a function on $\mathcal{E}^\infty$ may be defined by

$$\mathbf{E}(\psi)\equiv\frac{\delta\psi}{\delta u}=\ell_\psi^*(1), \tag{15}$$

while in the coordinate form we have

$$\mathbf{E}(\psi)\equiv\frac{\delta\psi}{\delta u}=\sum_{s\geq 0}(-D_x)^s\Big(\frac{\partial\psi}{\partial u_s}\Big). \tag{16}$$

If the number of dependent variables is >1, say m, then $\mathbf{E}(\psi)$ is an m-component vector with the components

$$\mathbf{E}(\psi)^j=\sum_{s\geq 0}(-D_x)^s\Big(\frac{\partial\psi}{\partial u_s^j}\Big),\qquad j=1,\dots,m.$$

Let us mention two important properties of the Euler operator:

(1) $\mathbf{E}(\psi)=0$ if and only if $\psi=D_x(\varphi)$ for some φ;
(2) $\xi=\mathbf{E}(\psi)$ if and only if $\ell_\xi=\ell_\xi^*$.

1.4. **Conservation laws.** A function ψ is a *conserved current* for equation (1) if there exists another function ζ such that

$$D_t(\psi)=D_x(\zeta).$$

A conserved current is called *trivial* if $\psi=D_x(\varphi)$ for some function φ. Two currents are said to be *equivalent* if their difference is trivial and the equivalence class of currents is called a *conservation law* for (1). Denote the space of conservation laws by $\mathrm{cl}(\mathcal{E})$.

Let ψ be a conserved current. Then from Property (1) of the Euler operator it follows that the function $\xi=\mathbf{E}(\psi)$ depends on the corresponding conservation law only. It is called the *generating function* of this conservation law. Generating functions of conservation laws for equation (1) satisfy the equation

$$\ell_{\mathcal{E}}^*(\xi)=0, \tag{17}$$

where $\ell_{\mathcal{E}}$ is given by (9), but not all solutions of (17) are generating functions of conservation laws. From Property (2) it follows that ξ should satisfy

$$\ell_\xi=\ell_\xi^*. \tag{18}$$

Thus, to find a conservation law, one needs to solve equation (17) and check property (18).

Remark. Evolutionary vector fields act on generating functions of conservation laws and this action is expressed by the formula

$$L_{\mathbf{\ni}_\varphi}(\psi)=\mathbf{\ni}_\varphi(\psi)+\ell_\varphi^*(\psi), \tag{19}$$

where $\mathbf{\ni}_\varphi(\psi)$ is understood in the “usual” way.

1.5. **Nonlocal structures.** In many cases, the space of functions on $\mathcal{E}^\infty$ is not sufficient to describe the object arising naturally in applications (the reader will see examples in Section 3, see also [8]). One of the ways to extend this space is to introduce objects of the form $\int \varphi\,dx$ (or, as they are denoted sometimes, $D_x^{-1}(\varphi)$), where φ is a function. A formal procedure to define these objects is as follows.

Let us add to the space of functions new variables $w_1, \dots, w_l$ and vector fields

$$X = X_1 \frac{\partial}{\partial w_1} + \dots + X_l \frac{\partial}{\partial w_l}, \quad T = T_1 \frac{\partial}{\partial w_1} + \dots + T_l \frac{\partial}{\partial w_l}$$

satisfying the consistency conditions

$$[\bar{D}_x, \bar{D}_t] \equiv D_x(T) - D_t(X) + [X, T] = 0 \tag{20}$$

where

$$\bar{D}_x = D_x + X, \qquad \bar{D}_t = D_t + T$$

are extensions of the total derivatives to the "new" function space. The variables $w_1, \dots, w_l$ satisfy the system of differential equations

$$w_{i,x} = X_i(t, x, u, \dots, u_s, w), \tag{21}$$

$$w_{i,t} = T_i(t, x, u, \dots, u_s, w), \tag{22}$$

and compatibility conditions are fulfilled on $\mathcal{E}^\infty$ due to (20). We call $w_1, \dots, w_l$ the *nonlocal variables* associated to conditions (20).

The simplest nonlocal variables arise in the case when $l = 1$ and the functions $X = X_1$ and $T = T_1$ are independent of w. In this situation (20) reads

$$D_x(T) - D_t(X) = 0,$$

i.e., X satisfies (20) if and only if it is a conserved current. Moreover, the corresponding variable w is "really" nonlocal if and only if this current is nontrivial in the sense of Subsection 1.4. By (21), $w_x = X$, or $w = D_x^{-1}(X)$.

Note that any $\mathcal{C}$-differential operator $\Delta = \sum_{i,j} a_{ij} D_x^i D_t^j$ can be extended to the nonlocal setting by introducing the operator $\bar{\Delta} = \sum_{i,j} a_{ij} \bar{D}_x^i \bar{D}_t^j$. In particular, we can consider the equations

$$\bar{\ell}_{\mathcal{E}}(\varphi) = 0 \tag{23}$$

and

$$\bar{\ell}^*_{\mathcal{E}}(\xi) = 0 \tag{24}$$

(nonlocal analogues of (10) and (17), respectively), whose solutions are generating functions of *nonlocal symmetries* and *nonlocal conservation laws*.

1.6. **Hamiltonian operators.** Denote by $\mathrm{gf}(\mathcal{E})$ the space of generating functions of conservation laws, i.e., the set of solutions of equation (17) and consider a $\mathcal{C}$-differential operator H

$$H\colon \mathrm{gf}(\mathcal{E}) \to \mathrm{sym}(\mathcal{E}).$$

Then one may take the composition

$$H \circ \mathbf{E}\colon \mathrm{cl}(\mathcal{E}) \xrightarrow{\mathbf{E}} \mathrm{gf}(\mathcal{E}) \xrightarrow{H} \mathrm{sym}(\mathcal{E})$$

and, using the natural action of the space $\mathrm{sym}(\mathcal{E})$ on $\mathrm{cl}(\mathcal{E})$, define the bracket

$$\{\psi, \psi'\}_H = \ni_{H\mathbf{E}(\psi)}(\psi'), \tag{25}$$

where ψ and ψ' are conservation laws, which is called the *Poisson bracket* associated to the operator H. An operator $H\colon \mathrm{gf}(\mathcal{E}) \to \mathrm{sym}(\mathcal{E})$ is called a *Hamiltonian operator* (or *Hamiltonian structure*[3]) for equation (1) if

(1) the bracket $\{\psi, \psi'\}_H$ is skew-symmetric;
(2) it satisfies the Jacobi identity.

These two properties are reformulated in terms of the operator H in the following way:

(1) Property (1) is equivalent to
$$H + H^* = 0. \tag{26}$$
(2) Property (2) reads
$$[\mathfrak{Э}_{H(\xi)}, H] = \ell_{H(\xi)} \circ H + H \circ \ell^*_{H(\xi)} \tag{27}$$
for all ξ of the form $\xi = \mathbf{E}(\psi)$, i.e., such that $\ell_\xi = \ell^*_\xi$.

Two Hamiltonian operators H and H' are said to be *compatible* (or constitute a *pencil*, [10]) if $\lambda H + \lambda' H'$ is a Hamiltonian operator for all $\lambda, \lambda' \in \mathbb{R}$. Equations possessing such structures are called *bi-Hamiltonian.* If H and H' are two Hamiltonian operators, then, similar to (27), the condition of their compatibility is written in the form
$$[\mathfrak{Э}_{H(\xi)}, H'] + [\mathfrak{Э}_{H'(\xi)}, H] = \ell_{H(\xi)} \circ H' + H' \circ \ell^*_{H(\xi)} + \ell_{H'(\xi)} \circ H + H \circ \ell^*_{H'(\xi)}. \tag{28}$$

Remark. Let us rewrite condition (27) in a different form:
$$\mathfrak{Э}_{H(\xi)} \circ H - H \circ \mathfrak{Э}_{H(\xi)} = \ell_{H(\xi)} \circ H + H \circ \ell^*_{H(\xi)}, \tag{29}$$
or
$$(\mathfrak{Э}_{H(\xi)} - \ell_{H(\xi)}) \circ H = H \circ (\mathfrak{Э}_{H(\xi)} + H \circ \ell^*_{H(\xi)}).$$
But the first factor at the left-hand side is the action of $\mathfrak{Э}_{H(\xi)}$ on $\mathrm{sym}(\mathcal{E})$:
$$L_{\mathfrak{Э}_{H(\xi)}}(\varphi) = \{\mathfrak{Э}_{H(\xi)}, \varphi\} = (\mathfrak{Э}_{H(\xi)} - \ell_{H(\xi)})(\varphi),$$
while the second factor at the right-hand side is the action of $\mathfrak{Э}_{H(\xi)}$ of $\mathrm{gf}(\mathcal{E})$, see (19). Thus (27) is equivalent to
$$[L_{\mathfrak{Э}_{H(\xi)}}, H] = 0. \tag{30}$$

2. The method

Existence of a bi-Hamiltonian structure guarantees complete integrability of the equation under consideration. Indeed, the celebrated F. Magri scheme [10] leads to infinite number conservation laws in involution, i.e. such that $\{\psi_i, \psi_j\}_H = 0$. Moreover, Property (2) of Hamiltonian structures means that the map $\hat{H} = H \circ \mathbf{E}\colon \mathrm{cl}(\mathcal{E}) \to \mathrm{sym}(\mathcal{E})$ is a Lie algebra homomorphism:
$$\hat{H}(\{\psi, \psi'\}_H) = \{\hat{H}(\psi), \hat{H}(\psi')\},$$
where the bracket at the right-hand side is given by (11). This leads to the existence of infinite series of commuting symmetries which can be obtained by action of the *recursion operator* $R = H' \circ H^{-1}$ when the second factor makes sense.

We offer a very simple scheme that is inverse to Magri's one. Using techniques of [7], we start with a recursion operator and use the latter for construction of Hamiltonian structures.

[3]Though it would be better to call it a *Poisson structure.*

2.1. **Relation between $\mathcal{C}$-differential operators and Cartan forms.** Let $\Delta = \sum_{i,j} a_{ij} D_x^i D_t^j$ be a $\mathcal{C}$-differential operator on equation (1). Then the action of such an operator on Cartan forms (4) is well defined by

$$\Delta(\omega_s) = \sum_{i,j} a_{ij} \underbrace{D_x(\dots(D_x}_{i\text{ times}}(\underbrace{D_t(\dots(D_t}_{j\text{ times}}(\omega_s))\dots)))\dots)$$

due to the fact that $D_x \lrcorner\, \omega_\alpha = D_t \lrcorner\, \omega_\alpha = 0$ for any Cartan form ω_α.

In particular, we can define the map

$$\mathbf{e}\colon \mathcal{C}\mathrm{Diff}(\mathcal{E}) \to \mathcal{C}\Lambda^1(\mathcal{E}) \tag{31}$$

from the space of all $\mathcal{C}$-differential operators on our equation to the space of all Cartan forms by setting

$$\mathbf{e}(\Delta) = \Delta(\omega_0), \qquad \Delta \in \mathcal{C}\mathrm{Diff}(\mathcal{E}).$$

This map possesses the following properties

(1) It is epimorphic, i.e., any element $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ is of the form $\mathbf{e}(\Delta)$ for some $\Delta \in \mathcal{C}\mathrm{Diff}(\mathcal{E})$. Moreover, for any $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ the action $\varphi \mapsto \ni_\varphi \lrcorner\, \omega \equiv \Delta_\omega(\varphi)$ is a $\mathcal{C}$-differential operator and
$$\mathbf{e}(\Delta_\omega) = \omega.$$
(2) For any $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ and $\Delta \in \mathcal{C}\mathrm{Diff}(\mathcal{E})$ one has
$$\Delta(\omega) = \mathbf{e}(\Delta \circ \Delta_\omega).$$
(3) Consider a form $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ and let us define a vector-valued form
$$\ni_\omega = \sum_{s \geq o} D_x^s(\omega) \otimes \frac{\partial}{\partial u_s}.$$
Then
$$\mathbf{e}(\Delta \circ \nabla) = \ni_{\mathbf{e}(\nabla)} \lrcorner\, \mathbf{e}(\Delta).$$
(4) Finally, a $\mathcal{C}$-differential operator Δ is taken to zero by the map $\mathbf{e}$ if and only if it is divisable by $\ell_\mathcal{E}$ form the right:
$$\ker(\mathbf{e}) = \{\, \Delta = \nabla \circ \ell_\mathcal{E} \mid \nabla \in \mathcal{C}\mathrm{Diff}(\mathcal{E}) \,\}.$$

From Properties (2) and (4) it follows that for any $\mathcal{C}$-differential operator Δ a form $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ satisfies the equation

$$\Delta(\omega) = 0$$

if and only if

$$\Delta \circ \Delta_\omega = \nabla \circ \ell_\mathcal{E}$$

for some $\mathcal{C}$-differential operator ∇. In particular, for the equation

$$\ell_\mathcal{E}(\omega) = 0 \tag{32}$$

we obtain

$$\ell_\mathcal{E} \circ \Delta_\omega = \nabla \circ \ell_\mathcal{E}, \tag{33}$$

while the solutions of the equation

$$\ell_\mathcal{E}^*(\omega) = 0 \tag{34}$$

are described by

$$\ell_\mathcal{E}^* \circ \Delta_\omega = \nabla \circ \ell_\mathcal{E}. \tag{35}$$

In other words, we obtained the following

Theorem. *Let $\mathcal{E}$ be an equation of the form* (1). *Then*:

(1) *For any solution $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ of equation* (32), *the $\mathcal{C}$-differential operator Δ_ω takes* $\mathrm{sym}(\mathcal{E})$ *to* $\mathrm{sym}(\mathcal{E})$.

(2) *For any solution $\omega \in \mathcal{C}\Lambda^1(\mathcal{E})$ of equation* (34), *the $\mathcal{C}$-differential operator Δ_ω takes* $\mathrm{sym}(\mathcal{E})$ *to* $\mathrm{gf}(\mathcal{E})$.

This result is the base for the scheme described in the next subsection.

2.2. **General description of the scheme.** Using the last result, we suggest the following scheme to construct Hamiltonian structures:

Step 1: Solving the equation

$$D_t(\psi) = D_x(\zeta),$$

find a "sufficient" number[4] of pair-wise inequivalent conserved currents $\psi_1, \dots, \psi_l$ together with the correspondent functions $\zeta_1, \dots, \zeta_l$.

Step 2: Extend the space of functions on $\mathcal{E}^\infty$ by introducing nonlocal variables $w_1, \dots, w_l$ corresponding to the currents $\psi_1, \dots, \psi_l$:

$$w_{i,x} = \psi_i, \qquad i = 1, \dots, l,$$

or

$$w_i = D_x^{-1}(\psi_i), \qquad i = 1, \dots, l,$$

Step 3: In accordance with the previous step, extend the space $\mathcal{C}\Lambda^1(\mathcal{E})$ of Cartan forms on $\mathcal{E}^\infty$ with the forms

$$\theta^j = dw_j - \psi_j\, dx - \zeta_j\, dt.$$

Denote by $\mathcal{C}\Lambda^1(\bar{\mathcal{E}})$ the extended space.

Step 4: Solve the equation

$$\bar{\ell}_{\mathcal{E}}(\theta) = 0$$

for $\theta \in \mathcal{C}\Lambda^1(\bar{\mathcal{E}})$. Let $\theta_1, \dots, \theta_s$ be the solutions and $R_1 = \Delta_{\theta_1}, \dots, R_s = \Delta_{\theta_s}$ be the corresponding $\mathcal{C}$-differential operators, $R_i\colon \mathrm{sym}(\mathcal{E}) \to \mathrm{sym}(\mathcal{E})$.

Step 5: Solve the equation

$$\bar{\ell}^*_{\mathcal{E}}(\theta) = 0$$

for $\theta \in \mathcal{C}\Lambda^1(\bar{\mathcal{E}})$. Let $\bar{\theta}_1, \dots, \bar{\theta}_p$ be the solutions and $S_1 = \Delta_{\bar{\theta}_1}, \dots, S_p = \Delta_{\bar{\theta}_p}$ be the corresponding $\mathcal{C}$-differential operators, $S_i\colon \mathrm{sym}(\mathcal{E}) \to \mathrm{gf}(\mathcal{E})$.

Step 6: Solve the operator equations

$$R_\alpha = H_\alpha^\beta \circ S_\beta, \qquad \alpha = 1, \dots, s,\ \beta = 1, \dots, p,$$

with respect to unknown $\mathcal{C}$-differential operators $H_\alpha^\beta\colon \mathrm{gf}(\mathcal{E}) \to \mathrm{sym}(\mathcal{E})$. Let $H_1, \dots, H_q$ be different solutions.

Step 7: Solve the operator equations

$$H_\alpha = R_\beta \circ \bar{H}_\alpha^\beta, \qquad \alpha = 1, \dots, q,\ \beta = 1, \dots, s,$$

with respect to unknown $\mathcal{C}$-differential operators $\bar{H}_\alpha^\beta\colon \mathrm{gf}(\mathcal{E}) \to \mathrm{sym}(\mathcal{E})$. Let $H_1, \dots, H_{\bar{q}}$ be the set of all different solutions obtained in Steps 6 and 7.

Step 8: Using criteria (26) and (27), check whether the pairs (H_i, H_j), $1 \le i < j \le \bar{q}$, form Hamiltonian pencils.

[4]What is "sufficient" depends on the equation and on the number of Hamiltonian structures we intend to find. See Section 3 for illustrations.

Remark. Of course, Steps 4–7 contain redundant actions (though it is not quite important, since they are fulfilled by computer). In practice, it usually sufficient to find one nontrivial R at Step 4, one solution S at Step 5, and two solutions H_1 and H_2 at Steps 6 and 7.

Remark. Let us write down the conditions on S following from the fact that this operator is inverse to a Hamiltonian one. Let $S \circ H = \mathrm{id}$ and $\varphi = H(\xi)$, or $\xi = S(\varphi)$. Applying (29) to an arbitrary ξ', we obtain

$$\ni_{H(\xi)}(H(\xi')) - \ell_{H(\xi)}(H(\xi')) = H(\ni_{H(\xi)}(\xi') + \ell^*_{H(\xi)}(\xi')).$$

Using the above notation, it amounts to

$$S(\ni_\varphi(\varphi') - \ell_\varphi(\varphi')) = \ni_\varphi(S(\varphi')) + \ell_\varphi(S(\varphi')),$$

or

$$[S, \ni_\varphi] = S \circ \ell_\varphi + \ell^*_\varphi \circ S.$$

This equation, similar to the equation for Hamiltonian operators, can be rewritten as

$$[L_{\ni_\varphi}, S] = 0.$$

In addition, S must satisfy

$$S^* + S = 0.$$

2.3. **Computational formulas.** We present here a list of computational formulas, given in coordinates, needed for the realization of the above scheme. Let k be the order of equation (1) and m be the number of unknown functions.

Conserved currents. Let $\psi = \psi(t, x, u_0, \dots, u_s$. Then, in general, the corresponding ζ is a function of $t, x, u_0, \dots, u_{s+k-1}$ and the defining equation is

$$\frac{\partial \psi}{\partial t} + \sum_{i=0}^{s} \sum_{j=1}^{m} D_x^i(f^j) \frac{\partial \psi}{\partial u_i^j} = \frac{\partial \zeta}{\partial x} + \sum_{i=0}^{s+k-1} \sum_{j=1}^{m} u_{i+1}^j \frac{\partial \zeta}{\partial u_i^j}.$$

Nonlocal extensions. Let $\psi_1, \dots, \psi_l$ be the conserved currents found at Step 1 with the corresponding $\zeta_1, \dots, \zeta_l$ and associated nonlocal variables $w_1, \dots, w_l$. Then the extended total derivatives are

$$\bar{D}_x = \sum_{i \geq 0} \sum_{j=1}^{m} u_{i+1}^j \frac{\partial}{\partial u_i^j} + \sum_{j=1}^{l} \psi_j \frac{\partial}{\partial w_j},$$

$$\bar{D}_t = \sum_{i \geq 0} \sum_{j=1}^{m} D_x^i(f^j) \frac{\partial}{\partial u_i^j} + \sum_{j=1}^{l} \zeta_j \frac{\partial}{\partial w_j}.$$

The action of these operators on Cartan forms in the extended setting is given by the following formulas. For the forms

$$\omega_s^j = du_s^j - u_{s+1}^j\, dx - D_x^s(f^j)\, dt$$

one has

$$\bar{D}_x(\omega_s^j) = \omega_{s+1}^j, \qquad \bar{D}_t(\omega_s^j) = \bar{D}_x^s\Big(\sum_{i=0}^{k} \sum_{\alpha=1}^{m} \frac{\partial f^j}{\partial u_i^\alpha} \omega_i^\alpha\Big), \tag{36}$$

while for

$$\theta^j = dw_j - \psi_j\, dx - \zeta_j\, dt$$

we obtain

$$\bar{D}_x(\theta^j) = \sum_{i=0}^{s}\sum_{\alpha=1}^{m} \frac{\partial \psi_j}{\partial u_i^\alpha}\omega_i^\alpha, \qquad \bar{D}_t(\theta^j) = \bar{D}_x\Big(\sum_{i=0}^{s+k-1}\sum_{\alpha=1}^{m} \frac{\partial \zeta_j}{\partial u_i^\alpha}\omega_i^\alpha\Big). \tag{37}$$

For the forms θ^j the corresponding $\mathcal{C}$-differential operators are

$$\Delta_{\theta^j} = \sum_{i=0}^{s} \bar{D}_x^{-1} \circ \frac{\partial \psi_j}{\partial u_i^\alpha} \circ \bar{D}_x^i.$$

If $\varphi = \varphi(t, x, u_0, \dots, u_n, w_1, \dots, w_l)$ is a function in the extended setting, then its linearization is of the form

$$\ell_\varphi = \sum_{j=1}^{m}\Big(\sum_{i=0}^{n} \frac{\partial \varphi}{\partial u_i^j}\bar{D}_x^i + \sum_{\alpha=1}^{l} \frac{\partial \varphi}{\partial w_\alpha}\bar{D}_x^{-1} \circ \sum_{i=0}^{s} \frac{\partial \psi_\alpha}{\partial u_i^j}\bar{D}_x^i\Big).$$

On the nonlocal structure of recursion and Hamiltonian operators. Consider Step 4 and let $\theta = r_o\omega_o + \dots + r_1\omega_1 + r_0\omega_0 + r_1'\theta_1 + \dots + r_l'\theta_l$. Then solving the defining equation $\bar{\ell}_{\mathcal{E}}(\theta) = 0$ and using identities (36) and (37), one sees that the right-hand side of this equation is of the form

$$\sum_{\alpha\geq 0} c_\alpha\omega_\alpha + \sum_{\beta=1}^{l} \bar{\ell}_{\mathcal{E}}(r_\beta')\theta_\beta.$$

This means that the coefficients at the nonlocal terms of recursion operators satisfy the equations

$$\bar{\ell}_{\mathcal{E}}(r_\beta') = 0$$

and consequently are (*nonlocal*) *symmetries* of equation (1). Thus, in the chosen setting, any recursion operator is of the form

$$R = \underbrace{\sum_{\alpha=0}^{o} r_\alpha D_x^\alpha}_{\text{the local part}} + \underbrace{\sum_{\beta=1}^{l} \varphi_\beta \bar{D}_x^{-1} \circ \bar{\ell}_{\psi_\beta}}_{\text{the nonlocal part}},$$

where $\varphi_1, \dots, \varphi_l$ are symmetries and $\psi_1, \dots, \psi_l$ are conserved currents.

On the other hand, let H be a local Hamiltonian operator. Then the "next" operator will be of the form $H' = R \circ H$. Hence, its nonlocal part is

$$\sum_{\beta=1}^{l} \varphi_\beta \bar{D}_x^{-1} \circ \bar{\ell}_{\psi_\beta} \circ H.$$

But using (14) and (15) together with the fact that $H^* = -H$, we get

$$(\bar{\ell}_{\psi_\beta} \circ H)(\varphi)\cdot 1 = \varphi \cdot H^*(\bar{\ell}_{\psi_\beta}(1)) + \bar{D}_x(\eta) = -\varphi \cdot H(\mathbf{E}(\psi_\beta)) + \bar{D}_x(\eta)$$

for any function φ and some η. In other words, the action of $\bar{\ell}_{\psi_\beta} \circ H$ is equivalent, modulo total derivatives, to the multiplication by $\bar{\varphi}_\beta$, where $\bar{\varphi}_\beta$ is the symmetry corresponding to the current ψ_β by the Hamiltonian operator H. Thus we have

$$H' = \text{The local part } + \sum_{\beta=1}^{l} \varphi_\beta \bar{D}_x^{-1} \circ \bar{\varphi}_\beta,$$

both φ_β and $\bar{\varphi}_\beta$ being symmetries.

3. Applications

Below we consider two illustrative applications of the above scheme: to the classical KdV equation and to the Boussinesq equation. All computations were done by the computer system described in [7, Ch. X] and we present here the final results only.

3.1. The KdV equation.

Consider the KdV equation presented in the form

$$u_t = uu_x + u_{xxx}. \tag{38}$$

3.1.1. *Conserved currents.* We constructed five solutions of the equation

$$D_t(\psi) = D_x(\zeta).$$

They are

$$\psi_1 = u,$$
$$\zeta_1 = \frac{1}{2}u^2 + u_2;$$

$$\psi_2 = u^2,$$
$$\zeta_2 = \frac{2}{3}u^3 + 2uu_2 - u_1^2;$$

$$\psi_3 = u^3 - 3u_1^2,$$
$$\zeta_3 = \frac{3}{4}u^4 + 3u^2u_2 - 6uu_1^2 - 6u_1u_3 + 3u_2^2;$$

$$\psi_4 = u^4 - 12uu_1^2 + \frac{36}{5}u_2^2,$$
$$\zeta_4 = \frac{4}{5}u^5 + 4u^3u_2 - 18u^2u_1^2 - 24uu_1u_3 + \frac{98}{5}uu_2^2 + 12u_1^2u_2 + \frac{72}{5}u_2u_4 - \frac{36}{5}u_3^2;$$

$$\psi_5 = u^5 + 10u^3u_2 + 18u^2u_4 + \frac{108}{7}uu_6,$$
$$\begin{aligned}\zeta_5 = {} & \frac{5}{6}u^6 + 15u^4u_2 + 28u^3u_4 + 42u^2u_1u_3 + 84u^2u_2^2 + \frac{234}{7}u^2u_6 - 84uu_1^2u_2 \\ & + \frac{456}{7}uu_1u_5 + \frac{1872}{7}uu_2u_4 + \frac{1080}{7}uu_3^2 + \frac{108}{7}uu_8 + 21u_1^4 - \frac{396}{7}u_1^2u_4 \\ & - \frac{1080}{7}u_1u_2u_3 - \frac{108}{7}u_1u_7\frac{360}{7}u_2^3 + \frac{108}{7}u_2u_6.\end{aligned}$$

Let $w_1, \dots, w_5$ be nonlocal variables corresponding to these conserved currents.

We shall also use a *nonlocal* conservation law ψ_6:

$$\psi_6 = w_1, \qquad \zeta_6 = u_1 + \frac{1}{2}w_2.$$

The corresponding nonlocal variable w_6 may be understood as $\iint u\,dx$.

3.1.2. *Generating functions.* The generating functions corresponding the the conserved currents $\psi_1, \dots, \psi_5$ are

$$\xi_1 = 1,$$
$$\xi_2 = u,$$

$$\xi_3 = u^2 + 2u_2,$$
$$\xi_4 = u^3 + 6uu_2 + 3u_1^2 + \frac{18}{5}u_4,$$
$$\xi_5 = u^4 + 12u^2u_2 + 12uu_1^2 + \frac{72}{5}uu_4 + \frac{144}{5}u_1u_3 + \frac{108}{5}u_2^2 + \frac{216}{35}u_6.$$

3.1.3. *Recursion operators.* Using the nonlocal setting described above, we found five solutions to the equation $\bar{\ell}_{\mathcal{E}}(\theta) = 0$, i.e., recursion operators:

$$R_0 = \omega_0,$$
$$R_1 = \omega_2 + \frac{2}{3}u\omega_0 + \frac{1}{3}u_1\theta_1,$$
$$R_2 = \omega_4 + \frac{4}{3}u\omega_2 + 2u_1\omega_1 + \Big(\frac{4}{9}u^2 + \frac{4}{3}u_2\Big)\omega_0 + \frac{1}{3}\Big(uu_1 + u_3\Big)\theta_1 + \frac{1}{18}u_1\theta_2,$$
$$R_3 = \omega_6 + 2u\omega_4 + 5u_1\omega_3 + \Big(\frac{4}{3}u^2 + \frac{20}{3}u_2\Big)\omega_2 + \Big(4uu_1 + 5u_3\Big)\omega_1 + \Big(\frac{8}{27}u^3 + \frac{8}{3}uu_2 + 2u_1^2 + 2u_4\Big)\omega_0 + \Big(\frac{5}{18}u^2u_1 + \frac{5}{9}uu_3 + \frac{10}{9}u_1u_2 + \frac{1}{3}u_5\Big)\theta_1 + \frac{1}{18}\Big(uu_1 + u_3\Big)\theta_2 + \frac{1}{54}u_1\theta_3,$$
$$R_4 = \omega_8 + \frac{8}{3}\omega_6u + \frac{28}{3}\omega_5u_1 + \Big(\frac{8}{3}u^2 + \frac{56}{3}u_2\Big)\omega_4 + \Big(\frac{40}{3}uu_1 + \frac{70}{3}u_3\Big)\omega_3 + \Big(\frac{32}{27}u^3 + \frac{160}{9}uu_2 + \frac{122}{9}u_1^2 + \frac{56}{3}u_4\Big)\omega_2 + \Big(\frac{16}{3}u^2u_1 + \frac{40}{3}uu_3 + \frac{245}{9}u_1u_2 + \frac{28}{3}u_5\Big)\omega_1 + \Big(\frac{16}{81}u^4 + \frac{32}{9}u^2u_2 + \frac{16}{3}uu_1^2 + \frac{16}{3}uu_4 + \frac{40}{3}u_1u_3 + \frac{80}{9}u_2^2 + \frac{8}{3}u_6\Big)\omega_0 + \Big(\frac{35}{162}u^3u_1 + \frac{35}{54}u^2u_3 + \frac{70}{27}uu_1u_2 + \frac{7}{9}uu_5 + \frac{35}{54}u_1^3 + \frac{7}{3}u_1u_4 + \frac{35}{9}u_2u_3 + \frac{1}{3}u_7\Big)\theta_1 + \Big(\frac{5}{108}u^2u_1 + \frac{5}{54}uu_3 + \frac{5}{27}u_1u_2 + \frac{1}{18}u_5\Big)\theta_2 + \frac{1}{54}\Big(uu_1 + u_3\Big)\theta_3 + \frac{5}{648}u_1\theta_4$$

3.1.4. *S-operators.* We constructed five solutions of the equation $\bar{\ell}^*_{\mathcal{E}}(\theta) = 0$, i.e., found operators $S_i\colon \operatorname{sym}(\mathcal{E}) \to \operatorname{gf}(\mathcal{E})$, $i = 1, \dots, 5$:

$$S_1 = \theta_1,$$
$$S_2 = \omega_1 + \frac{1}{3}u\theta_1 + \frac{1}{6}\theta_2,$$
$$S_3 = \omega_3 + \frac{4}{3}u\omega_1 + u_1\omega_0 + \Big(\frac{1}{6}u^2 + \frac{1}{3}u_2\Big)\theta_1 + \frac{1}{18}u\theta_2 + \frac{1}{18}\theta_3,$$
$$S_4 = \omega_5 + 2u\omega_3 + 3u_1\omega_2 + \Big(\frac{4}{3}u^2 + \frac{10}{3}u_2\Big)\omega_1 + \Big(2uu_1 + \frac{5}{3}u_3\Big)\omega_0 + \Big(\frac{5}{54}u^3 + \frac{5}{9}uu_2 + \frac{5}{18}u_1^2 + \frac{1}{3}u_4\Big)\theta_1 + \Big(\frac{1}{36}u^2 + \frac{1}{18}u_2\Big)\theta_2 + \frac{1}{54}u\theta_3 + \frac{5}{216}\theta_4,$$
$$S_5 = \omega_7 + \frac{5}{2}u\omega_5 + \frac{41}{6}u_1\omega_4 + \Big(\frac{89}{36}u^2 + \frac{71}{6}u_2\Big)\omega_3 + \Big(\frac{151}{18}uu_1 + \frac{23}{2}u_3\Big)\omega_2$$

$$
\begin{aligned}
&+ \Big(\frac{349}{324}u^3 + \frac{167}{18}uu_2 + \frac{31}{6}u_1^2 + \frac{43}{6}u_4\Big)\omega_1 \\
&+ \Big(\frac{253}{108}u^2u_1 + \frac{73}{18}uu_3 + \frac{139}{18}u_1u_2 + \frac{13}{6}u_5\Big)\omega_0 \\
&+ \Big(\frac{35}{648}u^4 + \frac{35}{54}u^2u_2 + \frac{35}{54}uu_1^2 + \frac{7}{9}uu_4 + \frac{14}{9}u_1u_3 + \frac{7}{6}u_2^2 + \frac{1}{3}u_6\Big)\theta_1 \\
&+ \Big(\frac{5}{324}u^3 + \frac{5}{54}uu_2 + \frac{5}{108}u_1^2 + \frac{1}{18}u_4\Big)\theta_2 + \Big(\frac{1}{108}u^2 + \frac{1}{54}u_2\Big)\theta_3 \\
&+ \frac{5}{648}u\theta_4 + \frac{7}{648}\theta_5.
\end{aligned}
$$

3.1.5. *Remarks on actions.* Let us rewrite the operators R and S above in a more conventional way using the results of Subsection 2.3. The action corresponding to the forms ω_i is just D_x^i. The action corresponding to the nonlocal forms

$$\theta_i = dw_i - \psi_i\,dx - \zeta_i\,dt$$

is as follows

$$
\begin{aligned}
&\theta_1 \mapsto D_x^{-1}, \\
&\theta_2 \mapsto 2D_x^{-1} \circ u, \\
&\theta_3 \mapsto 3D_x^{-1} \circ \big(u^2 - 2u_1D_x\big), \\
&\theta_4 \mapsto 4D_x^{-1} \circ \Big(u^3 - 3u_1^2 - 6uu_1D_x + \frac{18}{5}u_2D_x^2\Big), \\
&\theta_5 \mapsto 5D_x^{-1} \circ \Big(u^4 + 6u^2u_2 + \frac{36}{5}uu_4 + \frac{108}{35}u_6 + 2u^3D_x^2 + \frac{18}{5}u^2D_x^4 + \frac{108}{35}uD_x^6\Big).
\end{aligned}
$$

Hence, the recursion operators are represented as

$$
\begin{aligned}
R_0 &= \mathrm{id}, \\
R_1 &= D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1D_x^{-1}, \\
R_2 &= D_x^4 + \frac{4}{3}uD_x^2 + 2u_1D_x + \Big(\frac{4}{9}u^2 + \frac{4}{3}u_2\Big) + \frac{1}{3}\Big(uu_1 + u_3\Big)D_x^{-1} + \frac{1}{9}u_1D_x^{-1} \circ u, \\
R_3 &= D_x^6 + 2uD_x^4 + 5u_1D_x^3 + \Big(\frac{4}{3}u^2 + \frac{20}{3}u_2\Big)D_x^2 + \big(4uu_1 + 5u_3\big)D_x \\
&+ \Big(\frac{8}{27}u^3 + \frac{8}{3}uu_2 + 2u_1^2 + 2u_4\Big) + \Big(\frac{5}{18}u^2u_1 + \frac{5}{9}uu_3 + \frac{10}{9}u_1u_2 + \frac{1}{3}u_5\Big)D_x^{-1} \\
&+ \frac{1}{18}\Big(uu_1 + u_3\Big)\theta_2 + \frac{1}{18}u_1D_x^{-1} \circ \Big(u^2 - 2u_1D_x\Big),
\end{aligned}
$$

etc., while the S-operators are

$$
\begin{aligned}
S_1 &= D_x^{-1}, \\
S_2 &= D_x + \frac{1}{3}uD_x^{-1} + \frac{1}{3}D_x^{-1} \circ u, \\
S_3 &= D_x^3 + \frac{4}{3}uD_x + u_1 + \Big(\frac{1}{6}u^2 + \frac{1}{3}u_2\Big)D_x^{-1} + \frac{1}{9}uD_x^{-1} \circ u \\
&+ \frac{1}{6}D_x^{-1} \circ \Big(u^2 - 2u_1D_x\Big),
\end{aligned}
$$

etc. Note that $R_i = R_1^i$.

3.1.6. *The first decomposition.* We found, among others, the following decompositions:

$$R_0 = H_1 \circ S_1,$$
$$R_1 = H_2 \circ S_1,$$

where

$$H_1 = D_x,$$
$$H_2 = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1.$$

3.1.7. *The second decomposition.* On the other hand,

$$H_2 = R_1 \circ H_1.$$

3.1.8. *Hamiltonian operators.* Thus

$$H_1 = D_x, \qquad H_2 = D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1$$

are the first two local Hamiltonian structures for the KdV equation.

3.2. **The Boussinesq equation.** We consider this equation as the system of the form

$$\begin{cases} u_t &= u_x v + uv_x + \sigma v_{xxx}, \\ v_t &= u_x + vv_x, \end{cases} \tag{39}$$

where σ is a real constant.

3.2.1. *Conserved currents.* We found and used the following conserved currents for equation (39)

$$\psi_1 = v,$$
$$\zeta_1 = u + \frac{1}{2}v^2;$$

$$\psi_2 = u,$$
$$\zeta_2 = \sigma v_2 + uv;$$

$$\psi_3 = uv,$$
$$\zeta_3 = \sigma vv_2 - \frac{1}{2}\sigma v_1^2 + \frac{1}{2}u^2 + uv^2;$$

$$\psi_4 = \sigma vv_2 + u^2 + uv^2,$$
$$\zeta_4 = 2\sigma uv_2 - \sigma u_1 v_1 + \sigma u_2 v + 2\sigma v^2 v_2 + 2u^2 v + uv^3;$$

$$\psi_5 = v\Big(4\sigma u_2 + \frac{3}{2}\sigma vv_2 + 3u^2 + uv^2\Big),$$
$$\zeta_5 = 4\sigma^2 vv_4 - 4\sigma^2 v_1 v_3 + 4\sigma^2 v_2^2 + 10\sigma uvv_2 - 5\sigma uv_1^2 + 2\sigma u_1^2 + 5\sigma u_1 vv_1 + \frac{11}{2}\sigma u_2 v^2 + \frac{5}{2}\sigma v^3 v_2 + u^3 + \frac{9}{2}u^2 v^2 + uv^4;$$

$$\psi_6 = 4\sigma^2 vv_4 + 4\sigma uu_2 + 10\sigma uvv_2 + 3\sigma u_2 v^2 + 2\sigma v^3 v_2 + 2u^3 + 6u^2 v^2 - 10\sigma^2 v_1^2 v_2 + uv^4,$$

$$
\begin{aligned}
\zeta_6 &= 4\sigma^2 uv_4 + 4\sigma^2 u_2 v_2 - 4\sigma^2 u_3 v_1 + 4\sigma^2 u_4 v + 7\sigma^2 v^2 v_4 + 10\sigma^2 vv_1 v_3 \\
&+ 20\sigma^2 vv_2^2 + 10\sigma u^2 v_2 + 2\sigma uu_1 v_1 + 14\sigma uu_2 v + 25\sigma uv^2 v_2 - 2\sigma u_1^2 v \\
&+ 5\sigma u_2 v^3 + 3\sigma v^4 v_2 + 6u^3 v + 8u^2 v^3 + uv^5;
\end{aligned}
$$

$$
\begin{aligned}
\psi_7 &= v\Big(16\sigma^2 u_4 + 10\sigma^2 vv_4 + 40\sigma uu_2 + 25\sigma uvv_2 + 20\sigma u_1^2 + 5\sigma u_2 v^2 \\
&+ \frac{5}{2}\sigma v^3 v_2 + 10u^3 + 10u^2 v^2 + uv^4\Big), \\
\zeta_7 &= 16\sigma^3 vv_6 - 16\sigma^3 v_1 v_5 + 16\sigma^3 v_2 v_4 - 8\sigma^3 v_3^2 + 56\sigma^2 uvv_4 - 56\sigma^2 uv_1 v_3 \\
&+ 28\sigma^2 uv_2^2 + 16\sigma^2 u_1 u_3 + 84\sigma^2 u_1 vv_3 - 28\sigma^2 u_1 v_1 v_2 - 8\sigma^2 u_2^2 \\
&+ 116\sigma^2 u_2 vv_2 - 44\sigma^2 u_2 v_1^2 + 44\sigma^2 u_3 vv_1 + 26\sigma^2 u_4 v^2 - 50\sigma^2 vv_1^2 v_2 \\
&+ \frac{25}{2}\sigma^2 v_1^4 + 70\sigma u^2 vv_2 - 35\sigma u^2 v_1^2 + 20\sigma uu_1^2 + 70\sigma uu_1 vv_1 + 65\sigma uu_2 v^2 \\
&+ 50\sigma uv^3 v_2 + 15\sigma u_1^2 v^2 + \frac{15}{2}\sigma u_2 v^4 + \frac{7}{2}\sigma v^5 v_2 + \frac{5}{2}u^4 + 20u^3 v^2 \\
&+ \frac{25}{2}u^2 v^4 + uv^6;
\end{aligned}
$$

$$
\begin{aligned}
\psi_8 &= 16\sigma^3 vv_6 + 16\sigma^2 uu_4 + 56\sigma^2 uvv_4 + 20\sigma^2 u_4 v^2 + \frac{15}{2}\sigma^2 v^3 v_4 \\
&+ \frac{75}{2}\sigma^2 v^2 v_2^2 + 20\sigma u^2 u_2 + 70\sigma u^2 vv_2 + 50\sigma uu_2 v^2 + 50\sigma uv^3 v_2 \\
&- 10\sigma u_1^2 v^2 + \frac{15}{2}\sigma u_2 v^4 + 3\sigma v^5 v_2 + 5u^4 + 30u^3 v^2 + 15u^2 v^4 + uv^6, \\
\zeta_8 &= 16\sigma^3 uv_6 + 16\sigma^3 u_4 v_2 - 16\sigma^3 u_5 v_1 + 32\sigma^3 u_6 v + 36\sigma^3 v^2 v_6 + 56\sigma^3 vv_1 v_5 \\
&+ 280\sigma^3 vv_2 v_4 + 168\sigma^3 vv_3^2 - 56\sigma^3 v_1^2 v_4 - 168\sigma^3 v_1 v_2 v_3 + 56\sigma^3 v_2^3 + 36\sigma^2 u^2 v_4 \\
&+ 64\sigma^2 uu_1 v_3 + 136\sigma^2 uu_2 v_2 + 24\sigma^2 uu_3 v_1 + 72\sigma^2 uu_4 v + 126\sigma^2 uv^2 v_4 \\
&+ 140\sigma^2 uvv_1 v_3 + 280\sigma^2 uvv_2^2 - 140\sigma^2 uv_1^2 v_2 - 64\sigma^2 u_1^2 v_2 - 8\sigma^2 u_1 u_2 v_1 \\
&- 16\sigma^2 u_1 u_3 v + \frac{65}{2}\sigma^2 u_1 v^2 v_3 - 205\sigma^2 u_1 vv_1 v_2 + 115\sigma^2 u_1 v_1^3 + 8\sigma^2 u_2^2 v \\
&+ \frac{435}{2}\sigma^2 u_2 v^2 v_2 - 115\sigma^2 u_2 vv_1^2 + \frac{115}{2}\sigma^2 u_3 v^2 v_1 + \frac{55}{2}\sigma^2 u_4 v^3 + 15\sigma^2 v^4 v_4 \\
&+ 100\sigma^2 v^3 v_2^2 + 40\sigma u^3 v_2 + 10\sigma u^2 u_1 v_1 + 90\sigma u^2 u_2 v + 210\sigma u^2 v^2 v_2 - 20\sigma uu_1^2 v \\
&+ 100\sigma uu_2 v^3 + \frac{175}{2}\sigma uv^4 v_2 - 20\sigma u_1^2 v^3 + 21\sigma u_2 v^5 + 4\sigma v^6 v_2 + 20u^4 v \\
&+ 50u^3 v^3 + 18u^2 v^5 + uv^7;
\end{aligned}
$$

$$
\begin{aligned}
\psi_9 &= v\Big(64\sigma^3 u_6 + 28\sigma^3 vv_6 + 56\sigma^3 v_2 v_4 + 224\sigma^2 uu_4 + 196\sigma^2 uvv_4 + 448\sigma^2 u_1 u_3 \\
&+ 280\sigma^2 u_1 vv_3 + 336\sigma^2 u_2^2 + 420\sigma^2 u_2 vv_2 + \frac{105}{8}\sigma^2 v^3 v_4 + \frac{175}{2}\sigma^2 v^2 v_2^2 \\
&+ 280\sigma u^2 u_2 + 245\sigma u^2 vv_2 + 280\sigma uu_1^2 + \frac{350}{3}\sigma uu_2 v^2 + \frac{175}{2}\sigma uv^3 v_2 - \frac{70}{3}\sigma u_1^2 v^2 \\
&+ \frac{21}{2}\sigma u_2 v^4 + \frac{7}{2}\sigma v^5 v_2 + 35u^4 + 70u^3 v^2 + 21u^2 v^4 + uv^6\Big),
\end{aligned}
$$

$$\begin{aligned}
\zeta_9 = {} & 64\sigma^4 vv_8 - 64\sigma^4 v_1v_7 + 64\sigma^4 v_2v_6 - 64\sigma^4 v_3v_5 + 32\sigma^4 v_4^2 + 288\sigma^3 uvv_6 \\
& - 288\sigma^3 uv_1v_5 + 288\sigma^3 uv_2v_4 - 144\sigma^3 uv_3^2 + 64\sigma^3 u_1u_5 + 664\sigma^3 u_1vv_5 \\
& - 376\sigma^3 u_1v_1v_4 + 144\sigma^3 u_1v_2v_3 - 64\sigma^3 u_2u_4 + 1352\sigma^3 u_2vv_4 - 976\sigma^3 u_2v_1v_3 \\
& + 416\sigma^3 u_2v_2^2 + 32\sigma^3 u_3^2 + 1392\sigma^3 u_3vv_3 - 416\sigma^3 u_3v_1v_2 + 1072\sigma^3 u_4vv_2 \\
& - 328\sigma^3 u_4v_1^2 + 328\sigma^3 u_5vv_1 + 92\sigma^3 u_6v^2 + 28\sigma^3 v^3v_6 + 168\sigma^3 v^2v_1v_5 \\
& + 896\sigma^3 v^2v_2v_4 + 308\sigma^3 v^2v_3^2 - 336\sigma^3 vv_1^2v_4 - 616\sigma^3 vv_1v_2v_3 + 392\sigma^3 vv_2^3 \\
& + 336\sigma^3 v_1^3v_3 - 196\sigma^3 v_1^2v_2^2 + 504\sigma^2 u^2vv_4 - 504\sigma^2 u^2v_1v_3 + 252\sigma^2 u^2v_2^2 \\
& + 224\sigma^2 uu_1u_3 + 1512\sigma^2 uu_1vv_3 - 504\sigma^2 uu_1v_1v_2 - 112\sigma^2 uu_2^2 + 1960\sigma^2 uu_2vv_2 \\
& - 728\sigma^2 uu_2v_1^2 + 728\sigma^2 uu_3vv_1 + 420\sigma^2 uu_4v^2 + \frac{938}{3}\sigma^2 uv^3v_4 + 630\sigma^2 uv^2v_1v_3 \\
& + 1260\sigma^2 uv^2v_2^2 - 1260\sigma^2 uvv_1^2v_2 + 315\sigma^2 uv_1^4 + 224\sigma^2 u_1^2u_2 + 1120\sigma^2 u_1^2vv_2 \\
& - 308\sigma^2 u_1^2v_1^2 + 1344\sigma^2 u_1u_2vv_1 + 532\sigma^2 u_1u_3v^2 + \frac{1015}{6}\sigma^2 u_1v^3v_3 \\
& + \frac{2205}{2}\sigma^2 u_1v^2v_1v_2 - 315\sigma^2 u_1vv_1^3 + 504\sigma^2 u_2^2v^2 + \frac{1295}{2}\sigma^2 u_2v^3v_2 \\
& + \frac{315}{2}\sigma^2 u_2v^2v_1^2 - \frac{105}{2}\sigma^2 u_3v^3v_1 + \frac{105}{8}\sigma^2 u_4v^4 + \frac{189}{8}\sigma^2 v^5v_4 + \frac{1575}{8}\sigma^2 v^4v_2^2 \\
& + 420\sigma u^3vv_2 - 210\sigma u^3v_1^2 + 140\sigma u^2u_1^2 + 630\sigma u^2u_1vv_1 \\
& + 525\sigma u^2u_2v^2 + \frac{1715}{3}\sigma u^2v^3v_2 + 210\sigma uu_1^2v^2 + \frac{1225}{6}\sigma uu_2v^4 + 140\sigma uv^5v_2 \\
& - \frac{245}{6}\sigma u_1^2v^4 + 14\sigma u_2v^6 + \frac{9}{2}\sigma v^7v_2 + 7u^5 + \frac{175}{2}u^4v^2 + 105u^3v^4 \\
& + \frac{49}{2}u^2v^6 + uv^8;
\end{aligned}$$

$$\begin{aligned}
\psi_{10} = {} & 64\sigma^4 vv_8 + 64\sigma^3 uu_6 + 288\sigma^3 uvv_6 + \frac{672}{5}\sigma^3 u_1vv_5 + 336\sigma^3 u_4vv_2 \\
& + \frac{56}{5}\sigma^3 u_6v^2 - \frac{28}{9}\sigma^3 v^3v_6 + 924\sigma^3 v^2v_2v_4 + \frac{1400}{3}\sigma^3 v^2v_3^2 + 112\sigma^2 u^2u_4 \\
& + 504\sigma^2 u^2vv_4 + 784\sigma^2 uu_1vv_3 + 1176\sigma^2 uu_2vv_2 + 196\sigma^2 uu_4v^2 + \frac{623}{3}\sigma^2 uv^3v_4 \\
& + 945\sigma^2 uv^2v_2^2 + 168\sigma^2 u_1u_3v^2 - \frac{280}{3}\sigma^2 u_1v^3v_3 + 420\sigma^2 u_2^2v^2 + 490\sigma^2 u_2v^3v_2 \\
& + \frac{105}{4}\sigma^2 u_4v^4 + 21\sigma^2 v^5v_4 + 175\sigma^2 v^4v_2^2 + \frac{280}{3}\sigma u^3u_2 + 420\sigma u^3vv_2 \\
& + 490\sigma u^2u_2v^2 + \frac{1960}{3}\sigma u^2v^3v_2 - 140\sigma uu_1^2v^2 + \frac{700}{3}\sigma uu_2v^4 + 140\sigma uv^5v_2 \\
& - \frac{140}{3}\sigma u_1^2v^4 + 14\sigma u_2v^6 + 4\sigma v^7v_2 + 14u^5 + 140u^4v^2 + 140u^3v^4 \\
& + 28u^2v^6 + uv^8,
\end{aligned}$$

$$\begin{aligned}
\zeta_{10} = {} & 64\sigma^4 uv_8 + 64\sigma^4 u_6v_2 - 64\sigma^4 u_7v_1 + 64\sigma^4 u_8v + \frac{376}{5}\sigma^4 v^2v_8 + \frac{2448}{5}\sigma^4 vv_1v_7 \\
& + \frac{10752}{5}\sigma^4 vv_2v_6 + \frac{17568}{5}\sigma^4 vv_3v_5 + \frac{11712}{5}\sigma^4 vv_4^2 - \frac{2448}{5}\sigma^4 v_1^2v_6
\end{aligned}$$

$$
\begin{aligned}
&- \frac{5856}{5}\sigma^4 v_1 v_2 v_5 - \frac{11712}{5}\sigma^4 v_1 v_3 v_4 + \frac{5856}{5}\sigma^4 v_2^2 v_4 + 176\sigma^3 u^2 v_6 + 384\sigma^3 u u_1 v_5 \\
&+ 960\sigma^3 u u_2 v_4 + 1280\sigma^3 u u_3 v_3 + 1184\sigma^3 u u_4 v_2 + 160\sigma^3 u u_5 v_1 + 352\sigma^3 u u_6 v \\
&+ \frac{2476}{5}\sigma^3 u v^2 v_6 + \frac{8008}{5}\sigma^3 u v v_1 v_5 + \frac{30464}{5}\sigma^3 u v v_2 v_4 + \frac{14448}{5}\sigma^3 u v v_3^2 \\
&- \frac{8008}{5}\sigma^3 u v_1^2 v_4 - \frac{14448}{5}\sigma^3 u v_1 v_2 v_3 + \frac{4816}{5}\sigma^3 u v_2^3 - \frac{1248}{5}\sigma^3 u_1^2 v_4 \\
&- \frac{2304}{5}\sigma^3 u_1 u_2 v_3 - \frac{4096}{5}\sigma^3 u_1 u_3 v_2 - \frac{144}{5}\sigma^3 u_1 u_4 v_1 - \frac{656}{5}\sigma^3 u_1 u_5 v \\
&+ \frac{2464}{15}\sigma^3 u_1 v^2 v_5 - \frac{10808}{15}\sigma^3 u_1 v v_1 v_4 + \frac{2576}{15}\sigma^3 u_1 v v_2 v_3 + \frac{34832}{15}\sigma^3 u_1 v_1^2 v_3 \\
&- \frac{4816}{5}\sigma^3 u_1 v_1 v_2^2 + \frac{2304}{5}\sigma^3 u_2^2 v_2 - \frac{512}{5}\sigma^3 u_2 u_3 v_1 + \frac{656}{5}\sigma^3 u_2 u_4 v \\
&+ \frac{3136}{3}\sigma^3 u_2 v^2 v_4 - \frac{20552}{15}\sigma^3 u_2 v v_1 v_3 + \frac{14336}{5}\sigma^3 u_2 v v_2^2 - 952\sigma^3 u_2 v_1^2 v_2 \\
&+ \frac{512}{5}\sigma^3 u_3^2 v + \frac{1316}{3}\sigma^3 u_3 v^2 v_3 + \frac{2464}{5}\sigma^3 u_3 v v_1 v_2 + \frac{2296}{15}\sigma^3 u_3 v_1^3 \\
&+ \frac{4256}{3}\sigma^3 u_4 v^2 v_2 - \frac{2296}{15}\sigma^3 u_4 v v_1^2 + \frac{1148}{15}\sigma^3 u_5 v^2 v_1 + \frac{364}{45}\sigma^3 u_6 v^3 + \frac{833}{36}\sigma^3 v^4 v_6 \\
&- \frac{371}{3}\sigma^3 v^3 v_1 v_5 + \frac{4417}{3}\sigma^3 v^3 v_2 v_4 + \frac{1757}{9}\sigma^3 v^3 v_3^2 + 371\sigma^3 v^2 v_1^2 v_4 \\
&+ 4081\sigma^3 v^2 v_1 v_2 v_3 + 2968\sigma^3 v^2 v_2^3 - 742\sigma^3 v v_1^3 v_3 - 2968\sigma^3 v v_1^2 v_2^2 + 742\sigma^3 v_1^4 v_2 \\
&+ \frac{616}{3}\sigma^2 u^3 v_4 + 448\sigma^2 u^2 u_1 v_3 + 952\sigma^2 u^2 u_2 v_2 + 168\sigma^2 u^2 u_3 v_1 + 616\sigma^2 u^2 u_4 v \\
&+ 1190\sigma^2 u^2 v^2 v_4 + 1652\sigma^2 u^2 v v_1 v_3 + 3304\sigma^2 u^2 v v_2^2 - 1652\sigma^2 u^2 v_1^2 v_2 \\
&- 112\sigma^2 u u_1^2 v_2 - 504\sigma^2 u u_1 u_2 v_1 + 168\sigma^2 u u_1 u_3 v + 903\sigma^2 u u_1 v^2 v_3 \\
&+ 1162\sigma^2 u u_1 v v_1 v_2 + 714\sigma^2 u u_1 v_1^3 + 504\sigma^2 u u_2^2 v + 4837\sigma^2 u u_2 v^2 v_2 \\
&- 714\sigma^2 u u_2 v v_1^2 + 357\sigma^2 u u_3 v^2 v_1 + \frac{1211}{3}\sigma^2 u u_4 v^3 + \frac{1869}{4}\sigma^2 u v^4 v_4 \\
&+ 2835\sigma^2 u v^3 v_2^2 + 112\sigma^2 u_1^3 v_1 + 168\sigma^2 u_1^2 u_2 v + 133\sigma^2 u_1^2 v^2 v_2 - 714\sigma^2 u_1^2 v v_1^2 \\
&+ 1071\sigma^2 u_1 u_2 v^2 v_1 - 28\sigma^2 u_1 u_3 v^3 - 210\sigma^2 u_1 v^4 v_3 + 763\sigma^2 u_2^2 v^3 + \frac{2205}{2}\sigma^2 u_2 v^4 v_2 \\
&+ \frac{189}{4}\sigma^2 u_4 v^5 + 35\sigma^2 v^6 v_4 + 350\sigma^2 v^5 v_2^2 + \frac{490}{3}\sigma u^4 v_2 + \frac{140}{3}\sigma u^3 u_1 v_1 \\
&+ \frac{1540}{3}\sigma u^3 u_2 v + 1470\sigma u^3 v^2 v_2 - 140\sigma u^2 u_1^2 v + \frac{3430}{3}\sigma u^2 u_2 v^3 + \frac{3920}{3}\sigma u^2 v^4 v_2 \\
&- \frac{980}{3}\sigma u u_1^2 v^3 + \frac{1120}{3}\sigma u u_2 v^5 + 210\sigma u v^6 v_2 - \frac{224}{3}\sigma u_1^2 v^5 + 18\sigma u_2 v^7 + 5\sigma v^8 v_2 \\
&+ 70u^5 v + 280u^4 v^3 + 196u^3 v^5 + 32u^2 v^7 + uv^9.
\end{aligned}
$$

3.2.2. *Generating functions.* The generating functions corresponding to the conserved currents $\psi_, \dots, \psi_{10}$ are

$\xi_1^u = 0,$
$\xi_1^v = 1;$

$\xi_2^u = 1,$

$$\xi_2^v = 0;$$

$$\xi_3^u = 4\sigma u_2 + 6\sigma v v_2 + 3\sigma v_1^2 + 3u^2 + 3uv^2,$$
$$\xi_3^v = 4\sigma v_2 + 6uv + v^3;$$

$$\xi_4^u = 4(2\sigma^2 v_4 + 5\sigma u v_2 + 5\sigma u_1 v_1 + 4\sigma u_2 v + 3\sigma v^2 v_2 + 3\sigma v v_1^2 + 3u^2 v + uv^3),$$
$$\xi_4^v = 8\sigma u_2 + 16\sigma v v_2 + 6\sigma v_1^2 + 6u^2 + 12uv^2 + v^4;$$

$$\xi_5^u = 16\sigma^2 u_4 + 40\sigma^2 v v_4 + 80\sigma^2 v_1 v_3 + 60\sigma^2 v_2^2 + 40\sigma u u_2 + 100\sigma u v v_2 + 50\sigma u v_1^2 + 20\sigma u_1^2 + 100\sigma u_1 v v_1 + 40\sigma u_2 v^2 + 20\sigma v^3 v_2 + 30\sigma v^2 v_1^2 + 10u^3 + 30u^2 v^2 + 5uv^4,$$
$$\xi_5^v = 16\sigma^2 v_4 + 40\sigma u v_2 + 40\sigma u_1 v_1 + 40\sigma u_2 v + 40\sigma v^2 v_2 + 30\sigma v v_1^2 + 30u^2 v + 20uv^3 + v^5;$$

$$\xi_6^u = 2(16\sigma^3 v_6 + 56\sigma^2 u v_4 + 112\sigma^2 u_1 v_3 + 168\sigma^2 u_2 v_2 + 112\sigma^2 u_3 v_1 + 48\sigma^2 u_4 v + 60\sigma^2 v^2 v_4 + 240\sigma^2 v v_1 v_3 + 180\sigma^2 v v_2^2 + 210\sigma^2 v_1^2 v_2 + 70\sigma u^2 v_2 + 140\sigma u u_1 v_1 + 120\sigma u u_2 v + 150\sigma u v^2 v_2 + 150\sigma u v v_1^2 + 60\sigma u_1^2 v + 150\sigma u_1 v^2 v_1 + 40\sigma u_2 v^3 + 15\sigma v^4 v_2 + 30\sigma v^3 v_1^2 + 30u^3 v + 30u^2 v^3 + 3uv^5),$$
$$\xi_6^v = 32\sigma^2 u_4 + 96\sigma^2 v v_4 + 160\sigma^2 v_1 v_3 + 120\sigma^2 v_2^2 + 80\sigma u u_2 + 240\sigma u v v_2 + 100\sigma u v_1^2 + 40\sigma u_1^2 + 240\sigma u_1 v v_1 + 120\sigma u_2 v^2 + 80\sigma v^3 v_2 + 90\sigma v^2 v_1^2 + 20u^3 + 90u^2 v^2 + 30uv^4 + v^6;$$

$$\xi_7^u = 64\sigma^3 u_6 + 224\sigma^3 v v_6 + 672\sigma^3 v_1 v_5 + 1344\sigma^3 v_2 v_4 + 784\sigma^3 v_3^2 + 224\sigma^2 u u_4 + 784\sigma^2 u v v_4 + 1568\sigma^2 u v_1 v_3 + 1176\sigma^2 u v_2^2 + 448\sigma^2 u_1 u_3 + 1568\sigma^2 u_1 v v_3 + 3024\sigma^2 u_1 v_1 v_2 + 336\sigma^2 u_2^2 + 2352\sigma^2 u_2 v v_2 + 1512\sigma^2 u_2 v_1^2 + 1568\sigma^2 u_3 v v_1 + 336\sigma^2 u_4 v^2 + 280\sigma^2 v^3 v_4 + 1680\sigma^2 v^2 v_1 v_3 + 1260\sigma^2 v^2 v_2^2 + 2940\sigma^2 v v_1^2 v_2 + 315\sigma^2 v_1^4 + 280\sigma u^2 u_2 + 980\sigma u^2 v v_2 + 490\sigma u^2 v_1^2 + 280\sigma u u_1^2 + 1960\sigma u u_1 v v_1 + 840\sigma u u_2 v^2 + 700\sigma u v^3 v_2 + 1050\sigma u v^2 v_1^2 + 420\sigma u_1^2 v^2 + 700\sigma u_1 v^3 v_1 + 140\sigma u_2 v^4 + 42\sigma v^5 v_2 + 105\sigma v^4 v_1^2 + 35u^4 + 210u^3 v^2 + 105u^2 v^4 + 7uv^6,$$
$$\xi_7^v = 64\sigma^3 v_6 + 224\sigma^2 u v_4 + 448\sigma^2 u_1 v_3 + 672\sigma^2 u_2 v_2 + 448\sigma^2 u_3 v_1 + 224\sigma^2 u_4 v + 336\sigma^2 v^2 v_4 + 1120\sigma^2 v v_1 v_3 + 840\sigma^2 v v_2^2 + 840\sigma^2 v_1^2 v_2 + 280\sigma u^2 v_2 + 560\sigma u u_1 v_1 + 560\sigma u u_2 v + 840\sigma u v^2 v_2 + 700\sigma u v v_1^2 + 280\sigma u_1^2 v + 840\sigma u_1 v^2 v_1 + 280\sigma u_2 v^3 + 140\sigma v^4 v_2 + 210\sigma v^3 v_1^2 + 140u^3 v + 210u^2 v^3 + 42uv^5 + v^7;$$

$$\xi_8^u = 64\sigma^3 u_6 + 224\sigma^3 v v_6 + 672\sigma^3 v_1 v_5 + 1344\sigma^3 v_2 v_4 + 784\sigma^3 v_3^2 + 224\sigma^2 u u_4 + 784\sigma^2 u v v_4 + 1568\sigma^2 u v_1 v_3 + 1176\sigma^2 u v_2^2 + 448\sigma^2 u_1 u_3 + 1568\sigma^2 u_1 v v_3 + 3024\sigma^2 u_1 v_1 v_2 + 336\sigma^2 u_2^2 + 2352\sigma^2 u_2 v v_2 + 1512\sigma^2 u_2 v_1^2 + 1568\sigma^2 u_3 v v_1 + 336\sigma^2 u_4 v^2 + 280\sigma^2 v^3 v_4 + 1680\sigma^2 v^2 v_1 v_3 + 1260\sigma^2 v^2 v_2^2 + 2940\sigma^2 v v_1^2 v_2$$

$$
\begin{aligned}
&+ 315\sigma^2 v_1^4 + 280\sigma u^2 u_2 + 980\sigma u^2 v v_2 + 490\sigma u^2 v_1^2 + 280\sigma u u_1^2 + 1960\sigma u u_1 v v_1 \\
&+ 840\sigma u u_2 v^2 + 700\sigma u v^3 v_2 + 1050\sigma u v^2 v_1^2 + 420\sigma u_1^2 v^2 + 700\sigma u_1 v^3 v_1 \\
&+ 140\sigma u_2 v^4 + 42\sigma v^5 v_2 + 105\sigma v^4 v_1^2 + 35u^4 + 210u^3 v^2 + 105u^2 v^4 + 7uv^6, \\
\xi_8^v &= 64\sigma^3 v_6 + 224\sigma^2 u v_4 + 448\sigma^2 u_1 v_3 + 672\sigma^2 u_2 v_2 + 448\sigma^2 u_3 v_1 + 224\sigma^2 u_4 v \\
&+ 336\sigma^2 v^2 v_4 + 1120\sigma^2 v v_1 v_3 + 840\sigma^2 v v_2^2 + 840\sigma^2 v_1^2 v_2 + 280\sigma u^2 v_2 + 560\sigma u u_1 v_1 \\
&+ 560\sigma u u_2 v + 840\sigma u v^2 v_2 + 700\sigma u v v_1^2 + 280\sigma u_1^2 v + 840\sigma u_1 v^2 v_1 + 280\sigma u_2 v^3 \\
&+ 140\sigma v^4 v_2 + 210\sigma v^3 v_1^2 + 140u^3 v + 210u^2 v^3 + 42uv^5 + v^7;
\end{aligned}
$$

$$
\begin{aligned}
\xi_9^u &= 8(16\sigma^4 v_8 + 72\sigma^3 u v_6 + 216\sigma^3 u_1 v_5 + 456\sigma^3 u_2 v_4 + 552\sigma^3 u_3 v_3 + 456\sigma^3 u_4 v_2 \\
&+ 216\sigma^3 u_5 v_1 + 64\sigma^3 u_6 v + 112\sigma^3 v^2 v_6 + 672\sigma^3 v v_1 v_5 + 1344\sigma^3 v v_2 v_4 + 784\sigma^3 v v_3^2 \\
&+ 882\sigma^3 v_1^2 v_4 + 2856\sigma^3 v_1 v_2 v_3 + 658\sigma^3 v_2^3 + 126\sigma^2 u^2 v_4 + 504\sigma^2 u u_1 v_3 \\
&+ 756\sigma^2 u u_2 v_2 + 504\sigma^2 u u_3 v_1 + 224\sigma^2 u u_4 v + 392\sigma^2 u v^2 v_4 + 1568\sigma^2 u v v_1 v_3 \\
&+ 1176\sigma^2 u v v_2^2 + 1407\sigma^2 u v_1^2 v_2 + 462\sigma^2 u_1^2 v_2 + 924\sigma^2 u_1 u_2 v_1 + 448\sigma^2 u_1 u_3 v \\
&+ 784\sigma^2 u_1 v^2 v_3 + 3024\sigma^2 u_1 v v_1 v_2 + 693\sigma^2 u_1 v_1^3 + 336\sigma^2 u_2^2 v + 1176\sigma^2 u_2 v^2 v_2 \\
&+ 1512\sigma^2 u_2 v v_1^2 + 784\sigma^2 u_3 v^2 v_1 + 112\sigma^2 u_4 v^3 + 70\sigma^2 v^4 v_4 + 560\sigma^2 v^3 v_1 v_3 \\
&+ 420\sigma^2 v^3 v_2^2 + 1470\sigma^2 v^2 v_1^2 v_2 + 315\sigma^2 v v_1^4 + 105\sigma u^3 v_2 + 315\sigma u^2 u_1 v_1 \\
&+ 280\sigma u^2 u_2 v + 490\sigma u^2 v^2 v_2 + 490\sigma u^2 v v_1^2 + 280\sigma u u_1^2 v + 980\sigma u u_1 v^2 v_1 \\
&+ 280\sigma u u_2 v^3 + 175\sigma u v^4 v_2 + 350\sigma u v^3 v_1^2 + 140\sigma u_1^2 v^3 + 175\sigma u_1 v^4 v_1 + 28\sigma u_2 v^5 \\
&+ 7\sigma v^6 v_2 + 21\sigma v^5 v_1^2 + 35u^4 v + 70u^3 v^3 + 21u^2 v^5 + uv^7), \\
\xi_9^v &= 128\sigma^3 u_6 + 512\sigma^3 v v_6 + 1344\sigma^3 v_1 v_5 + 2688\sigma^3 v_2 v_4 + 1568\sigma^3 v_3^2 + 448\sigma^2 u u_4 \\
&+ 1792\sigma^2 u v v_4 + 3136\sigma^2 u v_1 v_3 + 2352\sigma^2 u v_2^2 + 896\sigma^2 u_1 u_3 + 3584\sigma^2 u_1 v v_3 \\
&+ 6048\sigma^2 u_1 v_1 v_2 + 672\sigma^2 u_2^2 + 5376\sigma^2 u_2 v v_2 + 3024\sigma^2 u_2 v_1^2 + 3584\sigma^2 u_3 v v_1 \\
&+ 896\sigma^2 u_4 v^2 + 896\sigma^2 v^3 v_4 + 4480\sigma^2 v^2 v_1 v_3 + 3360\sigma^2 v^2 v_2^2 + 6720\sigma^2 v v_1^2 v_2 \\
&+ 630\sigma^2 v_1^4 + 560\sigma u^2 u_2 + 2240\sigma u^2 v v_2 + 980\sigma u^2 v_1^2 + 560\sigma u u_1^2 + 4480\sigma u u_1 v v_1 \\
&+ 2240\sigma u u_2 v^2 + 2240\sigma u v^3 v_2 + 2800\sigma u v^2 v_1^2 + 1120\sigma u_1^2 v^2 + 2240\sigma u_1 v^3 v_1 \\
&+ 560\sigma u_2 v^4 + 224\sigma v^5 v_2 + 420\sigma v^4 v_1^2 + 70u^4 + 560u^3 v^2 + 420u^2 v^4 + 56uv^6 + v^8;
\end{aligned}
$$

$$
\begin{aligned}
\xi_{10}^u &= 256\sigma^4 u_8 + 1152\sigma^4 v v_8 + 4608\sigma^4 v_1 v_7 + 11904\sigma^4 v_2 v_6 + 19584\sigma^4 v_3 v_5 \\
&+ 11712\sigma^4 v_4^2 + 1152\sigma^3 u u_6 + 5184\sigma^3 u v v_6 + 15552\sigma^3 u v_1 v_5 + 31488\sigma^3 u v_2 v_4 \\
&+ 18528\sigma^3 u v_3^2 + 3456\sigma^3 u_1 u_5 + 15552\sigma^3 u_1 v v_5 + 44160\sigma^3 u_1 v_1 v_4 \\
&+ 73536\sigma^3 u_1 v_2 v_3 + 7296\sigma^3 u_2 u_4 + 32832\sigma^3 u_2 v v_4 + 76224\sigma^3 u_2 v_1 v_3 \\
&+ 53472\sigma^3 u_2 v_2^2 + 4416\sigma^3 u_3^2 + 39744\sigma^3 u_3 v v_3 + 77568\sigma^3 u_3 v_1 v_2 + 32832\sigma^3 u_4 v v_2 \\
&+ 22752\sigma^3 u_4 v_1^2 + 15552\sigma^3 u_5 v v_1 + 2304\sigma^3 u_6 v^2 + 2688\sigma^3 v^3 v_6 + 24192\sigma^3 v^2 v_1 v_5 \\
&+ 48384\sigma^3 v^2 v_2 v_4 + 28224\sigma^3 v^2 v_3^2 + 63504\sigma^3 v v_1^2 v_4 + 205632\sigma^3 v v_1 v_2 v_3 \\
&+ 47376\sigma^3 v v_2^3 + 46368\sigma^3 v_1^3 v_3 + 92232\sigma^3 v_1^2 v_2^2 + 2016\sigma^2 u^2 u_4 + 9072\sigma^2 u^2 v v_4 \\
&+ 18144\sigma^2 u^2 v_1 v_3 + 13608\sigma^2 u^2 v_2^2 + 8064\sigma^2 u u_1 u_3 + 36288\sigma^2 u u_1 v v_3
\end{aligned}
$$

$$
\begin{aligned}
&+ 69216\sigma^2 uu_1v_1v_2 + 6048\sigma^2 uu_2^2 + 54432\sigma^2 uu_2vv_2 + 34608\sigma^2 uu_2v_1^2\\
&+ 36288\sigma^2 uu_3vv_1 + 8064\sigma^2 uu_4v^2 + 9408\sigma^2 uv^3v_4 + 56448\sigma^2 uv^2v_1v_3\\
&+ 42336\sigma^2 uv^2v_2^2 + 101304\sigma^2 uvv_1^2v_2 + 11214\sigma^2 uv_1^4 + 7392\sigma^2 u_1^2u_2\\
&+ 33264\sigma^2 u_1^2vv_2 + 24024\sigma^2 u_1^2v_1^2 + 66528\sigma^2 u_1u_2vv_1 + 16128\sigma^2 u_1u_3v^2\\
&+ 18816\sigma^2 u_1v^3v_3 + 108864\sigma^2 u_1v^2v_1v_2 + 49896\sigma^2 u_1vv_1^3 + 12096\sigma^2 u_2^2v^2\\
&+ 28224\sigma^2 u_2v^3v_2 + 54432\sigma^2 u_2v^2v_1^2 + 18816\sigma^2 u_3v^3v_1 + 2016\sigma^2 u_4v^4\\
&+ 1008\sigma^2 v^5v_4 + 10080\sigma^2 v^4v_1v_3 + 7560\sigma^2 v^4v_2^2 + 35280\sigma^2 v^3v_1^2v_2 + 11340\sigma^2 v^2v_1^4\\
&+ 1680\sigma u^3u_2 + 7560\sigma u^3vv_2 + 3780\sigma u^3v_1^2 + 2520\sigma u^2u_1^2 + 22680\sigma u^2u_1vv_1\\
&+ 10080\sigma u^2u_2v^2 + 11760\sigma u^2v^3v_2 + 17640\sigma u^2v^2v_1^2 + 10080\sigma uu_1^2v^2\\
&+ 23520\sigma uu_1v^3v_1 + 5040\sigma uu_2v^4 + 2520\sigma uv^5v_2 + 6300\sigma uv^4v_1^2 + 2520\sigma u_1^2v^4\\
&+ 2520\sigma u_1v^5v_1 + 336\sigma u_2v^6 + 72\sigma v^7v_2 + 252\sigma v^6v_1^2 + 126u^5\\
&+ 1260u^4v^2 + 1260u^3v^4 + 252u^2v^6 + 9uv^8,
\end{aligned}
$$

$$
\begin{aligned}
\xi^v_{10} &= 256\sigma^4 v_8 + 1152\sigma^3 uv_6 + 3456\sigma^3 u_1v_5 + 7296\sigma^3 u_2v_4 + 8832\sigma^3 u_3v_3\\
&+ 7296\sigma^3 u_4v_2 + 3456\sigma^3 u_5v_1 + 1152\sigma^3 u_6v + 2304\sigma^3 v^2v_6 + 12096\sigma^3 vv_1v_5\\
&+ 24192\sigma^3 vv_2v_4 + 14112\sigma^3 vv_3^2 + 14112\sigma^3 v_1^2v_4 + 45696\sigma^3 v_1v_2v_3 + 10528\sigma^3 v_2^3\\
&+ 2016\sigma^2 u^2v_4 + 8064\sigma^2 uu_1v_3 + 12096\sigma^2 uu_2v_2 + 8064\sigma^2 uu_3v_1 + 4032\sigma^2 uu_4v\\
&+ 8064\sigma^2 uv^2v_4 + 28224\sigma^2 uvv_1v_3 + 21168\sigma^2 uvv_2^2 + 22512\sigma^2 uv_1^2v_2\\
&+ 7392\sigma^2 u_1^2v_2 + 14784\sigma^2 u_1u_2v_1 + 8064\sigma^2 u_1u_3v + 16128\sigma^2 u_1v^2v_3\\
&+ 54432\sigma^2 u_1vv_1v_2 + 11088\sigma^2 u_1v_1^3 + 6048\sigma^2 u_2^2v + 24192\sigma^2 u_2v^2v_2\\
&+ 27216\sigma^2 u_2vv_1^2 + 16128\sigma^2 u_3v^2v_1 + 2688\sigma^2 u_4v^3 + 2016\sigma^2 v^4v_4\\
&+ 13440\sigma^2 v^3v_1v_3 + 10080\sigma^2 v^3v_2^2 + 30240\sigma^2 v^2v_1^2v_2 + 5670\sigma^2 vv_1^4 + 1680\sigma u^3v_2\\
&+ 5040\sigma u^2u_1v_1 + 5040\sigma u^2u_2v + 10080\sigma u^2v^2v_2 + 8820\sigma u^2vv_1^2 + 5040\sigma uu_1^2v\\
&+ 20160\sigma uu_1v^2v_1 + 6720\sigma uu_2v^3 + 5040\sigma uv^4v_2 + 8400\sigma uv^3v_1^2 + 3360\sigma u_1^2v^3\\
&+ 5040\sigma u_1v^4v_1 + 1008\sigma u_2v^5 + 336\sigma v^6v_2 + 756\sigma v^5v_1^2 + 630u^4v + 1680u^3v^3\\
&+ 756u^2v^5 + 72uv^7 + v^9.
\end{aligned}
$$

3.2.3. *Recursion operators.*

$$
\begin{aligned}
R^u_0 &= \omega_0,\\
R^v_0 &= \tau_0;
\end{aligned}
$$

$$
\begin{aligned}
R^u_1 &= v\omega_0 + 2\sigma\tau_2 + 2u\tau_0 + u_1\theta_1,\\
R^v_1 &= 2\omega_0 + v\tau_0 + v_1\theta_1;
\end{aligned}
$$

$$
\begin{aligned}
R^u_2 &= 4\sigma\omega_2 + (4u + v^2)\omega_0 + 4\sigma v\tau_2 + 6\sigma v_1\tau_1 + (6\sigma v_2 + 4uv)\tau_0\\
&\quad + 2(\sigma v_3 + uv_1 + u_1v)\theta_1 + 2u_1\theta_2,\\
R^v_2 &= 4v\omega_0 + 4\sigma\tau_2 + (4u + v^2)\tau_0 + 2(u_1 + vv_1)\theta_1 + 2v_1\theta_2;
\end{aligned}
$$

$$
R^u_3 = 12\sigma v\omega_2 + 20\sigma v_1\omega_1 + (16\sigma v_2 + 12uv + v^3)\omega_0 + 8\sigma^2\tau_4 + (16\sigma u + 6\sigma v^2)\tau_2
$$

$$
\begin{aligned}
&+ (4\sigma u_1 + 18\sigma v v_1)\tau_1 + (16\sigma u_2 + 18\sigma v v_2 + 12\sigma v_1^2 + 8u^2 + 6uv^2)\tau_0\\
&+ (4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 + 6uu_1 + 6uvv_1 + 3u_1 v^2)\theta_1\\
&+ (4\sigma v_3 + 4uv_1 + 4u_1 v)\theta_2 + 2u_1\theta_3,\\
R_3^v &= 8\sigma\omega_2 + (8u + 6v^2)\omega_0 + \sigma v\tau_2 + 16\sigma v_1\tau_1 + (12\sigma v_2 + 12uv + v^3)\tau_0\\
&+ (4\sigma v_3 + 6uv_1 + 6u_1 v + 3v^2 v_1)\theta_1 + (4u_1 + 4vv_1)\theta_2 + 2v_1\theta_3;
\end{aligned}
$$

$$
\begin{aligned}
R_4^u &= 16\sigma^2\omega_4 + (32\sigma u + 24\sigma v^2)\omega_2 + (48\sigma u_1 + 80\sigma v v_1)\omega_1\\
&+ (32\sigma u_2 + 64\sigma v v_2 + 44\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)\omega_0 + 32\sigma^2 v\tau_4 + 80\sigma^2 v_1\tau_3\\
&+ (112\sigma^2 v_2 + 64\sigma uv + 8\sigma v^3)\tau_2 + (88\sigma^2 v_3 + 96\sigma uv_1 + 94\sigma u_1 v + 36\sigma v^2 v_1)\tau_1\\
&+ (40\sigma^2 v_4 + 80\sigma uv_2 + 106\sigma u_1 v_1 + 64\sigma u_2 v + 36\sigma v^2 v_2 + 48\sigma v v_1^2 + 32u^2 v\\
&+ 8uv^3)\tau_0 + (8\sigma^2 v_5 + 20\sigma uv_3 + 40\sigma u_1 v_2 + 36\sigma u_2 v_1 + 16\sigma u_3 v + 12\sigma v^2 v_3\\
&+ 48\sigma v v_1 v_2 + 12\sigma v_1^3 + 12u^2 v_1 + 24uu_1 v + 12uv^2 v_1 + 4u_1 v^3)\theta_1\\
&+ (8\sigma u_3 + 12\sigma v v_3 + 24\sigma v_1 v_2 + 12uu_1 + 12uvv_1 + 6u_1 v^2)\theta_2\\
&+ (4\sigma v_3 + 4uv_1 + 4u_1 v)\theta_3 + 2u_1\theta_4,\\
R_4^v &= 32\sigma v\omega_2 + 48\sigma v_1\omega_1 + (32\sigma v_2 + 32uv + 8v^3)\omega_0 + 16\sigma^2\tau_4 + (32\sigma u + 24\sigma v^2)\tau_2\\
&+ (48\sigma u_1 + 62\sigma v v_1)\tau_1 + (32\sigma u_2 + 48\sigma v v_2 + 30\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)\tau_0\\
&+ (8\sigma u_3 + 16\sigma v v_3 + 28\sigma v_1 v_2 + 12uu_1 + 24uvv_1 + 12u_1 v^2 + 4v^3 v_1)\theta_1\\
&+ (12uv_1 + 12u_1 v + 6v^2 v_1 + \sigma v_3)\theta_2 + (4u_1 + 4vv_1)\theta_3 + 2v_1\theta_4;
\end{aligned}
$$

$$
\begin{aligned}
R_5^u &= 80\sigma^2 v\omega_4 + 224\sigma^2 v_1\omega_3 + (320\sigma^2 v_2 + 160\sigma uv + 40\sigma v^3)\omega_2\\
&+ (240\sigma^2 v_3 + 256\sigma uv_1 + 232\sigma u_1 v + 200\sigma v^2 v_1)\omega_1\\
&+ (96\sigma^2 v_4 + 192\sigma uv_2 + 264\sigma u_1 v_1 + 160\sigma u_2 v + 160\sigma v^2 v_2 + 220\sigma v v_1^2\\
&+ 80u^2 v + 40uv^3 + v^5)\omega_0 + 32\sigma^3\tau_6 + (96\sigma^2 u + 80\sigma^2 v^2)\tau_4\\
&+ (240\sigma^2 u_1 + 400\sigma^2 v v_1)\tau_3 + (320\sigma^2 u_2 + 560\sigma^2 v v_2 + 408\sigma^2 v_1^2 + 96\sigma u^2\\
&+ 160\sigma uv^2 + 10\sigma v^4)\tau_2 + (240\sigma^2 u_3 + 436\sigma^2 v v_3 + 864\sigma^2 v_1 v_2 + 288\sigma uu_1\\
&+ 476\sigma uvv_1 + 233\sigma u_1 v^2 + 60\sigma v^3 v_1)\tau_1 + (96\sigma^2 u_4 + 200\sigma^2 v v_4 + 492\sigma^2 v_1 v_3\\
&+ 336\sigma^2 v_2^2 + 192\sigma uu_2 + 400\sigma uvv_2 + 276\sigma uv_1^2 + 136\sigma u_1^2 + 530\sigma u_1 v v_1\\
&+ 160\sigma u_2 v^2 + 60\sigma v^3 v_2 + 120\sigma v^2 v_1^2 + 32u^3 + 80u^2 v^2 + 10uv^4)\tau_0\\
&+ (16\sigma^2 u_5 + 40\sigma^2 v v_5 + 120\sigma^2 v_1 v_4 + 200\sigma^2 v_2 v_3 + 40\sigma uu_3 + 100\sigma uvv_3\\
&+ 200\sigma uv_1 v_2 + 80\sigma u_1 u_2 + 200\sigma u_1 v v_2 + 150\sigma u_1 v_1^2 + 180\sigma u_2 v v_1 + 40\sigma u_3 v^2\\
&+ 20\sigma v^3 v_3 + 120\sigma v^2 v_1 v_2 + 60\sigma v v_1^3 + 30u^2 u_1 + 60u^2 v v_1 + 60uu_1 v^2\\
&+ 20uv^3 v_1 + 5u_1 v^4)\theta_1 + (16\sigma^2 v_5 + 40\sigma uv_3 + 80\sigma u_1 v_2 + 72\sigma u_2 v_1 + 32\sigma u_3 v\\
&+ 24\sigma v^2 v_3 + 96\sigma v v_1 v_2 + 24\sigma v_1^3 + 24u^2 v_1 + 48uu_1 v + 24uv^2 v_1 + 8u_1 v^3)\theta_2\\
&+ (8\sigma u_3 + 12\sigma v v_3 + 24\sigma v_1 v_2 + 12uu_1 + 12uvv_1 + 6u_1 v^2)\theta_3\\
&+ (4\sigma v_3 + 4uv_1 + 4u_1 v)\theta_4 + 2\theta_5 u_1,\\
R_5^v &= 32\sigma^2\omega_4 + (64\sigma u + 80\sigma v^2)\omega_2 + (96\sigma u_1 + 232\sigma v v_1)\omega_1 + (64\sigma u_2 + 160\sigma v v_2
\end{aligned}
$$

$+ 112\sigma v_1^2 + 32u^2 + 80uv^2 + 10v^4)\omega_0 + 80\sigma^2 v\tau_4 + 176\sigma^2 v_1\tau_3 + (224\sigma^2 v_2$
$+ 160\sigma uv + 40\sigma v^3)\tau_2 + (176\sigma^2 v_3 + 224\sigma uv_1 + 236\sigma u_1 v + 153\sigma v^2 v_1)\tau_1$
$+ (80\sigma^2 v_4 + 160\sigma uv_2 + 228\sigma u_1 v_1 + 160\sigma u_2 v + 120\sigma v^2 v_2 + 150\sigma vv_1^2$
$+ 80u^2 v + 40uv^3 + v^5)\tau_0 + (16\sigma^2 v_5 + 40\sigma uv_3 + 80\sigma u_1 v_2 + 80\sigma u_2 v_1 + 40\sigma u_3 v$
$+ 40\sigma v^2 v_3 + 140\sigma vv_1 v_2 + 30\sigma v_1^3 + 30u^2 v_1 + 60uu_1 v + 60uv^2 v_1$
$+ 20u_1 v^3 + 5v^4 v_1)\theta_1 + (16\sigma u_3 + 32\sigma vv_3 + 56\sigma v_1 v_2 + 24uu_1 + 48uvv_1$
$+ 24u_1 v^2 + 8v^3 v_1)\theta_2 + (8\sigma v_3 + 12uv_1 + 12u_1 v + 6v^2 v_1)\theta_3 + (4u_1 + 4vv_1)\theta_4$
$+ 2v_1\theta_5.$

3.2.4. *S-operators.*

$S_1^u = v\theta_1 + 2\theta_2,$
$S_1^v = (u + 2\sigma)\tau_1 + \theta_3;$

$$S_2^u = 2\tau_1 + \Big(u + \frac{1}{2}v^2\Big)\theta_1 + v\theta_2 + \theta_3,$$
$$S_2^v = 2\sigma\omega_1 + \frac{3}{2}\sigma v\tau_1 + \frac{3}{2}\sigma v_1\tau_0 + (\sigma v_2 + uv)\theta_1 + \theta_2 u + \frac{1}{2}\theta_4;$$

$$S_3^u = \frac{8}{3}\sigma\omega_1 + + \frac{10}{3}\sigma v\tau_1 + 2\sigma v_1\tau_0 + \Big(\frac{4}{3}\sigma v_2 + 2uv + \frac{1}{3}v^3\Big)\theta_1 + \Big(\frac{4}{3}u + \frac{2}{3}v^2\Big)\theta_2 + \frac{2}{3}v\theta_3 + \frac{2}{3}\theta_4,$$
$$S_3^v = \frac{8}{3}\sigma v\omega_1 + 4\sigma v_1\omega_0 + \frac{8}{3}\tau_3\sigma^2 + \Big(\frac{16}{3}\sigma u + \frac{3}{2}\sigma v^2\Big)\tau_1 + \Big(\frac{8}{3}\sigma u_1 + 3\sigma vv_1\Big)\tau_0 + \Big(\frac{4}{3}\sigma u_2 + 2\sigma vv_2 + \sigma v_1^2 + u^2 + uv^2\Big)\theta_1 + \Big(\frac{4}{3}\sigma v_2 + \frac{4}{3}uv\Big)\theta_2 + \frac{2}{3}u\theta_3 + \frac{1}{3}\theta_5;$$

$S_4^u = v\theta_1 + 2\theta_2,$
$S_4^v = 2\sigma\tau_1 + u\theta_1 + \theta_3;$

$$S_5^u = 2\sigma\tau_1 + \Big(u + \frac{1}{2}v^2\Big)\theta_1 + v\theta_2 + \theta_3,$$
$$S_5^v = 2\sigma\omega_1 + \frac{3}{2}\sigma v\tau_1 + \frac{3}{2}\sigma v_1\tau_0 + (\sigma v_2 + uv)\theta_1 + u\theta_2 + \frac{1}{2}\theta_4;$$

$S_6^u = \theta_1,$
$S_6^v = \theta_2;$

$$S_7^u = \frac{32}{5}\sigma^2\omega_3 + \Big(\frac{56}{\sigma}u + \frac{66}{5}\sigma v^2\Big)\omega_1 + (8\sigma u_1 + 20\sigma vv_1)\omega_0 + \frac{72}{5}\sigma^2 v\tau_3 + \frac{104}{5}\sigma^2 v_1\tau_2 + \Big(24\sigma^2 v_2 + \frac{136}{5}\sigma uv + \frac{31}{5}\sigma v^3\Big)\tau_1 + \Big(\frac{56}{5}\sigma^2 v_3 + \frac{88}{5}\sigma uv_1 + 20\sigma u_1 v + 12\sigma v^2 v_1\Big)\tau_0 + \Big(\frac{16}{5}\sigma^2 v_4 + 8\sigma uv_2 + 8\sigma u_1 v_1 + 8\sigma u_2 v + 8\sigma v^2 v_2 + 6\sigma vv_1^2 + 6u^2 v + 4uv^3 + \frac{1}{5}v^5\Big)\theta_1 + \Big(\frac{16}{5}\sigma u_2 + \frac{32}{5}\sigma vv_2 + \frac{12}{5}\sigma v_1^2 + \frac{12}{5}u^2 + \frac{24}{5}uv^2 + \frac{2}{5}v^4\Big)\theta_2$$

$$
\begin{aligned}
&+ \Big(\frac{8}{5}\sigma v_2 + \frac{12}{5}uv + \frac{2}{5}v^3\Big)\theta_3 + \Big(\frac{4}{5}u + \frac{2}{5}v^2\Big)\theta_4 + \frac{2}{5}v\theta_5 + \frac{2}{5}\theta_6,\\
S_7^v = {}& \frac{64}{5}\sigma^2 v\omega_3 + 32\sigma^2 v_1\omega_2 + \Big(32\sigma^2 v_2 + \frac{112}{5}\sigma uv + 7\sigma v^3\Big)\omega_1\\
&+ \Big(16\sigma^2 v_3 + \frac{144}{5}\sigma uv_1 + 16\sigma u_1 v + 19\sigma v^2 v_1\Big)\omega_0 + \frac{32}{5}\sigma^3\tau_5\\
&+ \Big(\frac{96}{5}\sigma^2 u + 14\sigma^2 v^2\Big)\tau_3\\
&+ \Big(\frac{144}{5}\sigma^2 u_1 + 52\sigma^2 vv_1\Big)\tau_2\\
&+ \Big(\frac{176}{5}\sigma^2 u_2 + \frac{296}{5}\sigma^2 vv_2 + \frac{148}{5}\sigma^2 v_1^2 + \frac{96}{5}\sigma u^2 + \frac{128}{5}\sigma uv^2 + \frac{3}{2}\sigma v^4\Big)\tau_1\\
&+ \Big(\frac{64}{5}\sigma^2 u_3 + 28\sigma^2 vv_3 + \frac{272}{5}\sigma^2 v_1 v_2 + \frac{96}{5}\sigma uu_1 + 44\sigma uvv_1 + 21\sigma u_1 v^2\\
&+ 6\sigma v^3 v_1\Big)\tau_0 + \Big(\frac{16}{5}\sigma^2 u_4 + 8\sigma^2 vv_4 + 16\sigma^2 v_1 v_3 + 12\sigma^2 v_2^2 + 8\sigma uu_2 + 20\sigma uvv_2\\
&+ 10\sigma uv_1^2 + 4\sigma u_1^2 + 20\sigma u_1 vv_1 + 8\sigma u_2 v^2 + 4\sigma v^3 v_2 + 6\sigma v^2 v_1^2 + 2u^3 + 6u^2 v^2\Big)\theta_1\\
&+ uv^4 + \Big(\frac{16}{\sigma}{}^{2} v_4 + 8\sigma uv_2 + 8\sigma u_1 v_1 + \frac{32}{5}\sigma u_2 v + \frac{24}{5}\sigma v^2 v_2 + \frac{24}{5}\sigma vv_1^2 + \frac{24}{5}u^2 v\\
&+ \frac{8}{5}uv^3\Big)\theta_2 + \Big(\frac{8}{5}\sigma u_2 + \frac{12}{5}\sigma vv_2 + \frac{6}{5}\sigma v_1^2 + \frac{6}{5}u^2 + \frac{6}{5}uv^2\Big)\theta_3 + \Big(\frac{4}{5}\sigma v_2 + \frac{4}{5}uv\Big)\theta_4\\
&+ \frac{2}{5}u\theta_5 + \frac{1}{5}\theta_7;
\end{aligned}
$$

$$
\begin{aligned}
S_8^u = {}& \frac{80}{3}\sigma^2 v\omega_3 + \frac{160}{3}\sigma^2 v_1\omega_2 + \Big(\frac{160}{3}\sigma^2 v_2 + \frac{140}{3}\sigma uv + \frac{68}{3}\sigma v^3\Big)\omega_1\\
&+ \Big(\frac{80}{3}\sigma^2 v_3 + \frac{144}{3}\sigma uv_1 + \frac{100}{3}\sigma u_1 v + \frac{145}{3}\sigma v^2 v_1\Big)\omega_0 + \frac{32}{3}\sigma^3\tau_5\\
&+ \Big(32\sigma^2 u + \frac{56}{3}\sigma^2 v^2\Big)\tau_3 + \Big(288\sigma^2 u_1 + 104\sigma^2 vv_1\Big)\tau_2\\
&+ \Big(\frac{176}{3}\sigma^2 u_2 + \frac{356}{3}\sigma^2 vv_2 + \frac{148}{3}\sigma^2 v_1^2 + 32\sigma u^2 + 62\sigma uv^2 + \frac{23}{3}\sigma v^4\Big)\tau_1\\
&+ \Big(\frac{64}{3}\sigma^2 u_3 + 56\sigma^2 vv_3 + \frac{272}{3}\sigma^2 v_1 v_2 + 32\sigma uu_1 + 88\sigma uvv_1\\
&+ \frac{155}{3}\sigma u_1 v^2 + 20\sigma v^3 v_1\Big)\tau_0 + \Big(\frac{16}{3}\sigma^2 u_4 + 16\sigma^2 vv_4 + \frac{80}{3}\sigma^2 v_1 v_3 + 20\sigma^2 v_2^2\\
&+ \frac{40}{3}\sigma uu_2 + 40\sigma uvv_2 + \frac{50}{3}\sigma uv_1^2 + \frac{20}{3}\sigma u_1^2 + 40\sigma u_1 vv_1 + 20\sigma u_2 v^2\\
&+ \frac{40}{3}\sigma v^3 v_2 + 15\sigma v^2 v_1^2 + \frac{10}{3}u^3 + 15u^2 v^2 + 5uv^4 + v^6\Big)\theta_1\\
&+ \Big(\frac{16}{3}\sigma^2 v_4 + \frac{40}{3}\sigma uv_2 + \frac{40}{3}\sigma u_1 v_1 + \frac{40}{3}\sigma u_2 v + \frac{40}{3}\sigma v^2 v_2 + 10\sigma vv_1^2 + 10u^2 v\\
&+ \frac{20}{3}uv^3 + \frac{1}{3}v^5\Big)\theta_2 + \Big(\frac{8}{3}\sigma u_2 + \frac{16}{3}\sigma vv_2 + 2\sigma v_1^2 + 2u^2 + 4\theta_3 uv^2 + \frac{1}{3}v^4\Big)\theta_3\\
&+ \Big(\frac{4}{3}\sigma v_2 + 2uv + \frac{1}{3}v^3\Big)\theta_4 + \Big(\frac{2}{3}u + 2v^2\Big)\theta_5 + \frac{1}{3}v\theta_6 + \frac{1}{3}\theta_7,\\
S_8^v = {}& \frac{32}{3}\sigma^3\omega_5 + \Big(\frac{88}{3}\sigma^2 u + \frac{110}{3}\sigma^2 v^2\Big)\omega_3 + \Big(\frac{152}{3}\sigma^2 u_1 + \frac{452}{3}\sigma^2 vv_1\Big)\omega_2
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(56\sigma^2 u_2 + \frac{500}{3}\sigma^2 v v_2 + \frac{292}{3}\sigma^2 v_1^2 + \frac{82}{3}\sigma u^2 + 68\sigma u v^2 + \frac{35}{4}\sigma v^4\Big)\omega_1 \\
&+ \Big(24\sigma^2 u_3 + \frac{212}{3}\sigma^2 v v_3 + \frac{436}{3}\sigma^2 v_1 v_2 + 40\sigma u u_1 + \frac{352}{3}\sigma u v v_1 + \frac{155}{3}\sigma u_1 v^2 \\
&+ \frac{95}{3}\sigma v^3 v_1\Big)\omega_0 + \frac{88}{3}\sigma^3 v\tau_5 + \frac{248}{3}\sigma^3 v_1\tau_4 + \Big(136\sigma^3 v_2 + \frac{256}{3}\sigma^2 uv + \frac{305}{12}\sigma^2 v^3\Big)\tau_3 \\
&+ \Big(\frac{392}{3}\sigma^3 v_3 + \frac{464}{3}\sigma^2 u v_1 + \frac{460}{3}\sigma^2 u_1 v + \frac{495}{4}\sigma^2 v^2 v_1\Big)\tau_2 + \Big(88\sigma^3 v_4 + 176\sigma^2 u v_2 \\
&+ 200\sigma^2 u_1 v_1 + \frac{496}{3}\sigma^2 u_2 v + \frac{563}{4}\sigma^2 v^2 v_2 + \frac{319}{2}\sigma^2 v v_1^2 + 80\sigma u^2 v + \frac{127}{3}\sigma u v^3 \\
&+ \frac{3}{2}\sigma v^5\Big)\tau_1 + \Big(\frac{88}{3}\sigma^3 v_5 + 80\sigma^2 u v_3 + \frac{476}{3}\sigma^2 u_1 v_2 + \frac{464}{3}\sigma^2 u_2 v_1 + \frac{220}{3}\sigma^2 u_3 v \\
&+ \frac{305}{4}\sigma^2 v^2 v_3 + \frac{519}{2}\sigma^2 v v_1 v_2 + \frac{105}{2}\sigma^2 v_1^3 + 64\sigma u^2 v_1 + \frac{368}{3}\sigma u u_1 v + 110\sigma u v^2 v_1 \\
&+ 35\sigma u_1 v^3 + \frac{15}{2}\sigma v^4 v_1\Big)\tau_0 + \Big(\frac{16}{3}\sigma^3 v_6 + \frac{56}{3}\sigma^2 u v_4 + \frac{112}{3}\sigma^2 u_1 v_3 + 56\sigma^2 u_2 v_2 \\
&+ \frac{112}{3}\sigma^2 u_3 v_1 + 16\sigma^2 u_4 v + 20\sigma^2 v^2 v_4 + 80\sigma^2 v v_1 v_3 + 60\sigma^2 v v_2^2 + 70\sigma^2 v_1^2 v_2 \\
&+ \frac{70}{3}\sigma u^2 v_2 + \frac{140}{3}\sigma u u_1 v_1 + 40\sigma u u_2 v + 50\sigma u v^2 v_2 + 50\sigma u v v_1^2 + 20\sigma u_1^2 v \\
&+ 50\sigma u_1 v^2 v_1 + \frac{40}{3}\sigma u_2 v^3 + 5\sigma v^4 v_2 + 10\sigma v^3 v_1^2 + 10u^3 v + 10u^2 v^3 + uv^5\Big)\theta_1 \\
&+ \Big(\frac{16}{3}\sigma^2 u_4 + \frac{40}{3}\sigma^2 v v_4 + \frac{80}{3}\sigma^2 v_1 v_3 + 20\sigma^2 v_2^2 + \frac{40}{3}\sigma u u_2 + \frac{100}{3}\sigma u v v_2 + \frac{50}{3}\sigma u v_1^2 \\
&+ \frac{20}{3}\sigma u_1^2 + \frac{100}{3}\sigma u_1 v v_1 + \frac{40}{3}\sigma u_2 v^2 + \frac{20}{3}\sigma v^3 v_2 + 10\sigma v^2 v_1^2 + \frac{10}{3}u^3 + 10u^2 v^2 \\
&+ \frac{5}{3}uv^4\Big)\theta_2 + \Big(\frac{8}{3}\sigma^2 v_4 + \frac{20}{3}\sigma u v_2 + \frac{20}{3}\sigma u_1 v_1 + \frac{16}{3}\sigma u_2 v + 4\sigma v^2 v_2 + 4\sigma v v_1^2 \\
&+ 4u^2 v + \frac{4}{3}uv^3\Big)\theta_3 + \Big(\frac{4}{3}\sigma u_2 + 2\sigma v v_2 + \sigma v_1^2 + u^2 + uv^2\Big)\theta_4 + \Big(\frac{2}{3}\sigma v_2 + \frac{2}{3}uv\Big)\theta_5 \\
&+ \frac{1}{3}u\theta_6 + \frac{1}{6}\theta_8;
\end{aligned}
$$

$$
\begin{aligned}
S_9^u &= 6\sigma v\omega_1 + 6\sigma v_1\omega_0 + 4\tau_3\sigma^2 + \Big(8\sigma u + \frac{19}{4}\sigma v^2\Big)\tau_1 + \Big(4\sigma u_1 + 6\sigma v v_1\Big)\tau_0 \\
&+ \Big(2\sigma u_2 + 4\sigma v v_2 + \frac{3}{2}\sigma v_1^2 + \frac{3}{2}u^2 + 3uv^2 + \frac{1}{4}v^4\Big)\theta_1 + \Big(2\sigma v_2 + 3uv + \frac{1}{2}v^3\Big)\theta_2 \\
&+ \Big(u + \frac{1}{2}v^2\Big)\theta_3 + \frac{1}{2}v\theta_4 + \frac{1}{2}\theta_5, \\
S_9^v &= 4\sigma^2\omega_3 + \Big(7\sigma u + \frac{21}{4}\sigma v^2\Big)\omega_1 + \Big(5\sigma u_1 + \frac{19}{2}\sigma v v_1\Big)\omega_0 + 7\sigma^2 v\tau_3 + 13\sigma^2 v_1\tau_2 \\
&+ \Big(15\sigma^2 v_2 + 13\sigma u v + \frac{3}{2}\sigma v^3\Big)\tau_1 + \Big(7\sigma^2 v_3 + 11\sigma u v_1 + \frac{21}{2}\sigma u_1 v + \frac{9}{2}\sigma v^2 v_1\Big)\tau_0 \\
&+ (2\sigma^2 v_4 + 5\sigma u v_2 + 5\sigma u_1 v_1 + 4\sigma u_2 v + 3\sigma v^2 v_2 + 3\sigma v v_1^2 + 3u^2 v + uv^3)\theta_1 \\
&+ \Big(2\sigma u_2 + 3\sigma v v_2 + \frac{3}{2}\sigma v_1^2 + \frac{3}{2}u^2 + \frac{3}{2}uv^2\Big)\theta_2 + (\sigma v_2 + uv)\theta_3 + \frac{1}{2}u\theta_4 + \frac{1}{4}\theta_6.
\end{aligned}
$$

3.2.5. *Remarks on actions.* Similar to Subsection 3.1.5, we give here the operator presentations for R_i and S_i. In this case, these are 2×2-matrix operators, i.e.,

$$R_i = \begin{pmatrix} R_i^{11} & R_i^{12} \\ R_i^{21} & R_i^{22} \end{pmatrix}, \qquad S_i = \begin{pmatrix} S_i^{11} & S_i^{12} \\ S_i^{21} & S_i^{22} \end{pmatrix},$$

where

$$R_0^{11} = \mathrm{id},$$
$$R_0^{12} = 0,$$
$$R_0^{21} = 0,$$
$$R_0^{22} = \mathrm{id};$$

$$R_1^{11} = v,$$
$$R_1^{12} = 2\sigma D_x^2 + 2u + u_1 D_x^{-1},$$
$$R_1^{21} = 2,$$
$$R_1^{22} = v + v_1 D_x^{-1};$$

$$R_2^{11} =,$$
$$R_2^{12} = 4\sigma v D_x^2 + 6\sigma v_1 D_x + (6\sigma v_2 + 4uv) + 2(\sigma v_3 + uv_1 + u_1 v) D_x^{-1},$$
$$R_2^{21} = 4v + 2v_1 D_x^{-1},$$
$$R_2^{22} = 4\sigma D_x^2 + (4u + v^2) + 2(u_1 + vv_1) D_x^{-1};$$

$$\begin{aligned} R_3^{11} &= 12\sigma v D_x^2 + 20\sigma v_1 D_x + (16\sigma v_2 + 12uv + v^3) + (4\sigma v_3 + 4uv_1 + 4u_1 v) D_x^{-1} \\ &\quad + 2u_1 D_x^{-1} \circ v, \\ R_3^{12} &= 8\sigma^2 D_x^4 + (16\sigma u + 6\sigma v^2) D_x^2 + (4\sigma u_1 + 18\sigma v v_1) D_x \\ &\quad + (16\sigma u_2 + 18\sigma v v_2 + 12\sigma v_1^2 + 8u^2 + 6uv^2) \\ &\quad + (4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 + 6uu_1 + 6uvv_1 + 3u_1 v^2) D_x^{-1} + 2u_1 D_x^{-1} \circ u, \\ R_3^{21} &= 8\sigma D_x^2 + (8u + 6v^2) + (4u_1 + 4vv_1) D_x^{-1} + 2v_1 D_x^{-1} \circ v, \\ R_3^{22} &= \sigma v D_x^2 + 16\sigma v_1 D_x + (12\sigma v_2 + 12uv + v^3) \\ &\quad + (4\sigma v_3 + 6uv_1 + 6u_1 v + 3v^2 v_1) D_x^{-1} + 2v_1 D_x^{-1} \circ u, \end{aligned}$$

etc., and

$$S_1^{11} = 2D_x^{-1},$$
$$S_1^{12} = vD_x^{-1},$$
$$S_1^{21} = D_x^{-1} \circ v,$$
$$S_1^{22} = (u + 2\sigma) D_x + D_x^{-1} \circ u;$$

$$S_2^{11} = vD_x^{-1} + D_x^{-1} \circ v,$$
$$S_2^{12} = 2D_x + \left(u + \frac{1}{2}v^2\right) D_x^{-1} + D_x^{-1} \circ u,$$
$$S_2^{21} = 2\sigma D_x + \frac{3}{2}\sigma v D_x + uD_x^{-1} + D_x^{-1} \circ uv^2,$$

$$S_2^{22} = \frac{3}{2}\sigma v_1 + (\sigma v_2 + uv)D_x^{-1} + D_x^{-1} \circ \Big(\frac{1}{2}\sigma v_2 + uv\Big) + \frac{1}{2}D_x^{-1} \circ \sigma v D_x^2;$$

$$\begin{aligned}
S_3^{11} &= \frac{8}{3}\sigma D_x + \Big(\frac{4}{3}u + \frac{2}{3}v^2\Big)D_x^{-1} + \frac{2}{3}vD_x^{-1} \circ v + \frac{2}{3}D_x^{-1} \circ 2uv^2,\\
S_3^{12} &= \frac{10}{3}\sigma v D_x + 2\sigma v_1 + \Big(\frac{4}{3}\sigma v_2 + 2uv + \frac{1}{3}v^3\Big)D_x^{-1} + \frac{2}{3}vD_x^{-1} \circ u\\
&\quad + \frac{2}{3}D_x^{-1} \circ (\sigma v_2 + 2uv) + \frac{2}{3}D_x^{-1} \circ \sigma v D_x^2,\\
S_3^{21} &= \frac{8}{3}\sigma v D_x + 4\sigma v_1 + \Big(\frac{4}{3}\sigma v_2 + \frac{4}{3}uv\Big)D_x^{-1} + \frac{2}{3}uD_x^{-1} \circ 2uv^2 + \frac{1}{3}D_x^{-1} \circ v(6u_1v^2)\\
&\quad + \frac{4}{3}\sigma D_x^{-1} \circ vD_x^2,\\
S_3^{22} &= \frac{8}{3}\sigma^2 D_x^3 + \Big(\frac{16}{3}\sigma u + \frac{3}{2}\sigma v^2\Big)D_x + \Big(\frac{8}{3}\sigma u_1 + 3\sigma v v_1\Big)\\
&\quad + \Big(\frac{4}{3}\sigma u_2 + 2\sigma v v_2 + \sigma v_1^2 + u^2 + uv^2\Big)D_x^{-1} + \frac{2}{3}uD_x^{-1} \circ (\sigma v_2 + 2uv)\\
&\quad + \frac{2}{3}uD_x^{-1} \circ \sigma v D_x^2 + \frac{1}{3}D_x^{-1} \circ (4\sigma u_2 + 3\sigma v v_2 + 2uv) + \frac{1}{2}D_x^{-1} \circ \sigma v^2 D_x^2;
\end{aligned}$$

$$\begin{aligned}
S_4^{11} &= 2D_x^{-1},\\
S_4^{12} &= vD_x^{-1},\\
S_4^{21} &= D_x^{-1} \circ v,\\
S_4^{22} &= 2\sigma D_x + uD_x^{-1} + D_x^{-1} \circ u,
\end{aligned}$$

etc.

3.2.6. *The first decomposition.* We found the following decompositions:

$$\begin{aligned}
R_0 &= H_1 \circ S_1,\\
R_1 &= H_1 \circ S_2 = H_2 \circ S_1,\\
R_2 &= H_1 \circ S_3 = H_2 \circ S_2 = H_3 \circ S_1,
\end{aligned}$$

where

$$\begin{aligned}
H_1^u &= \omega_1,\\
H_1^v &= \tau_1;
\end{aligned}$$

$$\begin{aligned}
H_2^u &= v\omega_1 + v_1\omega_0 + 2\tau_1,\\
H_2^v &= 2\sigma\omega_3 + 2u\omega_1 + u_1\omega_0 + v\tau_1;
\end{aligned}$$

$$\begin{aligned}
H_3^u &= 4\sigma\omega_3 + (4u + v^2)\omega_1 + 2(u_1 + vv_1)\omega_0 + 4v\tau_1 + 2v_1\tau_0,\\
H_3^v &= 4\sigma v\omega_3 + 6\sigma v_1\omega_2 + 2(3\sigma v_2 + 2uv)\omega_1 + 2(\sigma v_3 + uv_1 + u_1v)\omega_0 + 4\sigma\tau_3\\
&\quad + (4u + v^2)\tau_1 + 2u_1\tau_0.
\end{aligned}$$

3.2.7. *The second decomposition.* One has

$$H_2 = R_1 \circ H_1, \qquad H_3 = R_1 \circ H_2 = R_2 \circ H_1.$$

Moreover, introducing formally the elements $\bar\theta_1, \dots, \bar\theta_4$ by

$$D_x(\bar\theta_1) = u_1\omega_0,$$

$$
\begin{aligned}
D_x(\bar\theta_2) &= v_1\tau_0,\\
D_x(\bar\theta_3) &= (\sigma v_3 + uv_1 + u_1v)\omega_0,\\
D_x(\bar\theta_4) &= (u_1 + vv_1)\tau_0,
\end{aligned}
$$

we can construct the nonlocal operators H_4 and H_5 of the form

$$
\begin{aligned}
H_4^u &= 12\sigma v\omega_3 + 16\sigma v_1\omega_2 + (12\sigma v_2 + 12uv + v^3)\omega_1\\
&\quad + (4\sigma v_3 + 8uv_1 + 6u_1v + 3v^2v_1)\omega_0 + 8\sigma\tau_3 + 2(4u + 3v^2)\tau_1 + 2(2u_1 + 3vv_1)\tau_0\\
&\quad - 2v_1\bar\theta_1 - 2v_1\bar\theta_2,\\
H_4^v &= 8\sigma^2\omega_5 + 2\sigma(8u + 3v^2)\omega_3 + 6\sigma(4u_1 + 3vv_1)\omega_2\\
&\quad + 2(8\sigma u_2 + 9\sigma vv_2 + 6\sigma v_1^2 + 4u^2 + 3uv^2)\omega_1\\
&\quad + (4\sigma u_3 + 6\sigma vv_3 + 12\sigma v_1v_2 + 8uu_1 + 6uvv_1 + 3u_1v^2)\omega_0 + 12\sigma v\tau_3 + 20\sigma v_1\tau_2\\
&\quad + (16\sigma v_2 + 12uv + v^3)\tau_1 + 2(2\sigma v_3 + 2uv_1 + 3u_1v)\tau_0 - 2u_1\bar\theta_1 - 2u_1\bar\theta_2;
\end{aligned}
$$

$$
\begin{aligned}
H_5^u &= 16\sigma^2\omega_5 + 8\sigma(4u + 3v^2)\omega_3 + 16\sigma(3u_1 + 4vv_1)\omega_2\\
&\quad + (32\sigma u_2 + 48\sigma vv_2 + 28\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)\omega_1\\
&\quad + 4(2\sigma u_3 + 4\sigma vv_3 + 8\sigma v_1v_2 + 4uu_1 + 8uvv_1 + 3u_1v^2 + v^3v_1)\omega_0\\
&\quad + 32\sigma v\tau_3 + 48\sigma v_1\tau_2 + 8(4\sigma v_2 + 4uv + v^3)\tau_1\\
&\quad + 4(2\sigma v_3 + 4uv_1 + 4u_1v + 3v^2v_1)\tau_0\\
&\quad - 4(u_1 + vv_1)\bar\theta_1 - 4(u_1 + vv_1)\bar\theta_2 - 4v_1\bar\theta_3 - 4v_1\bar\theta_4,\\
H_5^v &= 32\sigma^2v\omega_5 + 80\sigma^2v_1\omega_4 + 8\sigma(14\sigma v_2 + 8uv + v^3)\omega_3\\
&\quad + 4\sigma(22\sigma v_3 + 24uv_1 + 24u_1v + 9v^2v_1)\omega_2\\
&\quad + 4(10\sigma^2v_4 + 20\sigma uv_2 + 26\sigma u_1v_1 + 16\sigma u_2v + 9\sigma v^2v_2 + 12\sigma vv_1^2\\
&\quad + 8u^2v + 2uv^3)\omega_1 + 4(2\sigma^2v_5 + 6\sigma uv_3 + 11\sigma u_1v_2 + 9\sigma u_2v_1 + 4\sigma u_3v\\
&\quad + 3\sigma v^2v_3 + 12\sigma vv_1v_2 + 3\sigma v_1^3 + 4u^2v_1 + 8uu_1v + 3uv^2v_1 + u_1v^3)\omega_0\\
&\quad + 16\sigma^2\tau_5 + 8\sigma(4u + 3v^2)\tau_3 + 16\sigma(3u_1 + 5vv_1)\tau_2\\
&\quad + (32\sigma u_2 + 64\sigma vv_2 + 44\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)\tau_1\\
&\quad + 4(2\sigma u_3 + 4\sigma vv_3 + 6\sigma v_1v_2 + 4uu_1 + 4uvv_1 + 3u_1v^2)\tau_0\\
&\quad - 4(\sigma v_3 + uv_1 + u_1v)\bar\theta_1 - 4(\sigma v_3 + uv_1 + u_1v)\bar\theta_2 - 4u_1\bar\theta_3 - 4u_1\bar\theta_4.
\end{aligned}
$$

3.2.8. *Hamiltonian operators.* In the conventional representation the operators H_1, H_2 and H_3 are of the form

$$
H_1 = \begin{pmatrix} D_x & 0\\ 0 & D_x \end{pmatrix},
$$

$$
H_2 = \begin{pmatrix} vD_x + v_1 & 2D_x\\ 2\sigma D_x^3 + 2uD_x + u_1\omega_0 & vD_x \end{pmatrix},
$$

$$
H_3 = \begin{pmatrix} 4\sigma D_x^3 + (4u + v^2)D_x + 2(u_1 + vv_1), & 4vD_x + 2v_1,\\ 4\sigma vD_x^3 + 6\sigma v_1D_x^2 + 2(3\sigma v_2 + 2uv)D_x & 4\sigma D_x^3 + (4u + v^2)D_x\\ +2(\sigma v_3 + uv_1 + u_1v) & +2u_1 \end{pmatrix}
$$

For nonlocal operators

$$H_4 = \begin{pmatrix} H_4^{11} & H_4^{12} \\ H_4^{21} & H_4^{22} \end{pmatrix}, \qquad H_5 = \begin{pmatrix} H_5^{11} & H_5^{12} \\ H_5^{21} & H_5^{22} \end{pmatrix}$$

one has

$$\begin{aligned}
H_4^{11} &= 12\sigma v D_x^3 + 16\sigma v_1 D_x^2 + (12\sigma v_2 + 12uv + v^3)\omega_1 \\
&\quad + (4\sigma v_3 + 8uv_1 + 6u_1 v + 3v^2 v_1) - 2v_1 D_x^{-1} \circ u_1, \\
H_4^{12} &= 8\sigma D_x^3 + 2(4u + 3v^2)D_x + 2(2u_1 + 3vv_1) - 2v_1 D_x^{-1} \circ v_1, \\
H_4^{21} &= 8\sigma^2 D_x^5 + 2\sigma(8u + 3v^2)D_x^3 + 6\sigma(4u_1 + 3vv_1)D_x^2 \\
&\quad + 2(8\sigma u_2 + 9\sigma v v_2 + 6\sigma v_1^2 + 4u^2 + 3uv^2)D_x \\
&\quad + (4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 + 8uu_1 + 6uvv_1 + 3u_1 v^2) - 2u_1 D_x^{-1} \circ u_1, \\
H_4^{22} &= 12\sigma v D_x^3 + 20\sigma v_1 D_x^2 + (16\sigma v_2 + 12uv + v^3)D_x + 2(2\sigma v_3 + 2uv_1 + 3u_1 v) \\
&\quad - 2u_1 D_x^{-1} \circ v_1;
\end{aligned}$$

$$\begin{aligned}
H_5^{11} &= 16\sigma^2 D_x^5 + 8\sigma(4u + 3v^2)D_x^3 + 16\sigma(3u_1 + 4vv_1)D_x^2 \\
&\quad + (32\sigma u_2 + 48\sigma v v_2 + 28\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)D_x \\
&\quad + 4(2\sigma u_3 + 4\sigma v v_3 + 8\sigma v_1 v_2 + 4uu_1 + 8uvv_1 + 3u_1 v^2 + v^3 v_1) \\
&\quad - 4(u_1 + vv_1)D_x^{-1} \circ u_1 - 4v_1 D_x^{-1} \circ (\sigma v_3 + uv_1 + u_1 v), \\
H_5^{12} &= 32\sigma v D_x^3 + 48\sigma v_1 D_x^2 + 8(4\sigma v_2 + 4uv + v^3)D_x \\
&\quad + 4(2\sigma v_3 + 4uv_1 + 4u_1 v + 3v^2 v_1) - 4(u_1 + vv_1)D_x^{-1} \circ v_1 \\
&\quad - 4u_1 D_x^{-1} \circ (u_1 + vv_1), \\
H_5^{21} &= 32\sigma^2 v D_x^5 + 80\sigma^2 v_1 D_x^4 + 8\sigma(14\sigma v_2 + 8uv + v^3)D_x^3 \\
&\quad + 4\sigma(22\sigma v_3 + 24uv_1 + 24u_1 v + 9v^2 v_1)D_x^2 \\
&\quad + 4(10\sigma^2 v_4 + 20\sigma u v_2 + 26\sigma u_1 v_1 + 16\sigma u_2 v + 9\sigma v^2 v_2 + 12\sigma v v_1^2 \\
&\quad + 8u^2 v + 2uv^3)D_x + 4(2\sigma^2 v_5 + 6\sigma u v_3 + 11\sigma u_1 v_2 + 9\sigma u_2 v_1 + 4\sigma u_3 v \\
&\quad + 3\sigma v^2 v_3 + 12\sigma v v_1 v_2 + 3\sigma v_1^3 + 4u^2 v_1 + 8uu_1 v + 3uv^2 v_1 + u_1 v^3) \\
&\quad - 4(\sigma v_3 + uv_1 + u_1 v)D_x^{-1} \circ u_1) - 4u_1 D_x^{-1} \circ (\sigma v_3 + uv_1 + u_1 v), \\
H_5^{22} &= 16\sigma^2 D_x^5 + 8\sigma(4u + 3v^2)D_x^3 + 16\sigma(3u_1 + 5vv_1)D_x^2 \\
&\quad + (32\sigma u_2 + 64\sigma v v_2 + 44\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)D_x \\
&\quad + 4(2\sigma u_3 + 4\sigma v v_3 + 6\sigma v_1 v_2 + 4uu_1 + 4uvv_1 + 3u_1 v^2) \\
&\quad - 4(\sigma v_3 + uv_1 + u_1 v)D_x^{-1} \circ v_1 - 4u_1 D_x^{-1} \circ (u_1 + vv_1).
\end{aligned}$$

4. Concluding remarks

We introduced here a practical method of constructing Hamiltonian structures for evolution equations and systems of such equations. The method was tested on two known examples and demonstrated good results.

The quite extensive results in the second application (the Boussinesq equation) demonstrate clearly how smoothly the algorithmic procedures work in the construction of conservation laws, symmetries, recursion operators, generating functions, S-operators, and Hamiltonian structures.

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