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# Approximations for stop-loss reinsurance premiums

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# Approximations for stop-loss reinsurance premiums

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### Abstract

Various approximations of stop-loss reinsurance premiums are described in literature. For a wide variety of claim size distributions and retention levels, such approximations are compared in this paper to each other, as well as to a quantitative criterion. For the aggregate claims two models are used, both involving various model parameters. In the first model the claims are simply independent, while a certain dependence structure is assumed in the second model. A relatively simple rule of thumb is formulated for choosing the best approximation for either model. This approximation satisfies the aforementioned criterion. Finally, by comparing the two models, it is demonstrated that a small degree of dependence between the claims already has a substantial effect on the stop-loss premiums. The difference can run up to a factor 500.

*Keywords:* Individual model; Aggregate claims; Stop-loss premium; Dependent claims.

*Mathematics Subject Classification:* 62E17, 62P05

## 1 Introduction

In a nonlife insurance portfolio the aggregate claims during a reference period are given by the sum $S$ of the claims of the individual risks. Both for insurance and reinsurance companies, it is interesting to determine the distribution of $S$, since through this distribution various types of premiums can be calculated. Unfortunately, the distribution function of $S$ often is intractable. Simulation might be the solution, but there are many situations where this is too time-consuming. However, some moments of $S$ are relatively easily obtained and these suffice to obtain various approximations of the distribution function. With the latter, premiums can be approximated as well.

The most popular premium principle for reinsurance is the stop-loss principle with the premium $E(S-a)^+$ for various retentions $a$ (where $(x)^+ = \max(0,x)$). In literature, there are quite a few papers that describe a certain approximation of the distribution of $S$

and its accuracy (e.g. Seal (1977), Pentikäinen (1977) and Gendron and Crépeau (1989)). More recently, Chaubey et al. (1998) introduced some new approximations. The first goal of this paper is to find out which approximation of the stop-loss premium is best. In all the aforementioned papers the approximations of the distribution are investigated, whereas we investigate approximations of the stop-loss premiums. More importantly, we study a broad variety of claim size distributions, but nevertheless succeed in formulating a relatively simple rule of thumb for choosing an accurate solution, often being the best approximation. This rule of thumb is not limited to models with independent claims, but it can also be used for models with dependent claims. The fact that claims can be dependent is often ignored in insurance and reinsurance analyses. The second goal of our paper is to quantify the error that is made when the dependence of the claims is ignored. Albers (1999) has already shown for the simple case of normally distributed claim sizes that a small fraction of dependent claims already leads to very large errors (stop-loss premiums which are off by a factor 2 to 6). The present paper complements Albers (1999) since we also quantify this error for the harder but more important case of positively skewed claim sizes such as lognormal ones.

In order to achieve our first goal, we have to thoroughly compare the accuracy of the various approximations of the stop-loss premium. This thorough comparison requires an extensive investigation design. We have compared 5 approximations (normal power (NP), Edgeworth, gamma, inverse Gaussian (IG), gamma-IG) of the stop-loss premium at 10 retention levels. They are both compared to each other and to a quantitative criterion. The expected number of claims varied from 5 to 80 at a constant portfolio size of 10,000 contracts. Furthermore, 23 claim size distributions from 4 different families (gamma, IG, lognormal, Pareto) are selected, with skewness ranging from 0.1 to 14. This design is described in Section 2.

In Section 3, the claims are assumed to be independent. The extensiveness of the design allows us to obtain insight into the factors that determine the accuracy of the approximations. Having obtained this insight, the formulation of a rule of thumb providing an accurate approximation is the next step. Here 'accurate' means that the approximation meets a given criterion. Then in Section 4 a local dependence model is introduced, which is the nonlife insurance analogon of the local dependence model in Albers (1999). This model covers to first order many possible models for small degrees of dependence. For this general model of dependent claims, the accuracy of the approximations at various dependence levels is investigated. The expected number of dependent claims ranges from 1% to 5% of the expected total number of claims. Again, the main factors determining the accuracy are identified. A slightly modified rule of thumb provides an accurate approximation in the case of dependent claims. In Section 5 the results obtained are used to illustrate the difference between the stop-loss premiums of both models. They clearly show that ignoring dependence causes large errors. The conclusions are formulated in Section 6.

# 2 Preliminaries

As announced in the introduction, we shall first give an overview of the approximations examined and the claim size distributions selected. Subsequently the retention levels at which the approximations are investigated will be introduced and explained. At the end of this section, the aforementioned criterion will be clarified.

## 2.1 Approximations

There exist various approximations for the distribution of the total claim amount. The most well-known ones are the normal approximation and its refinements (normal power (NP) approximation and the Edgeworth expansions), the gamma approximation and the Esscher approximation. Moreover, Chaubey et al. (1998) have introduced the inverse Gaussian (IG) approximation and the gamma-IG approximation. In the present paper, the second order Edgeworth expansion is used, rather than the first order one or even the normal approximation itself, since the latter are known to be inaccurate. Furthermore, the Esscher approximation is not included as it requires the existence of the moment generating function of the claim size distribution. For lognormal or Pareto distributions, this function does not exist.

The basic principle of the remaining approximations is the same. The density $f_S(s)$ (or distribution function $P(S \leq s)$) is approximated by a function that uses the mean ($\mu_S$), variance ($\sigma_S^2$), skewness ($\kappa_{3S} = \tilde{\mu}_{3S}/\sigma_S^3$) and for some approximations (Edgeworth and gamma-IG) even the kurtosis ($\kappa_{4S} = \tilde{\mu}_{4S}/\sigma_S^4 - 3$) of $S$. Here, we introduce the notation $\tilde{\mu}_{jS}$ for the $j^{th}$ central moment of $S$. For all but the Edgeworth approximation, this approximating function is a density function or a distribution function with the same mean, variance, skewness and kurtosis as $S$. The Edgeworth expansion is formally not a distribution function. Nevertheless, it can be used to approximate interesting functionals (such as the stop-loss premium) of the distribution of $S$, just like the other approximations.

Next we give an overview of the distribution functions or the density functions of the approximations that we shall use in this paper.

**NP** The NP approximation extends the normal approximation with a correction for the positive skewness of the total claim amount (see e.g. Pentikäinen (1977)).

$$P\left(\frac{S-\mu_S}{\sigma_S} \leq z\right) \approx \mathbf{\Phi}\left(\sqrt{\frac{9}{\kappa_{3S}^2} + \frac{6z}{\kappa_{3S}} + 1} - \frac{3}{\kappa_{3S}}\right),$$

where $\mathbf{\Phi}(x)$ is the standard normal distribution function.

**Edgeworth** The second order Edgeworth expansion extends the normal approximation with corrections for both the skewness and the kurtosis of the total claim amount (see e.g. Feller (1971)).

$$P\left(\frac{S-\mu_S}{\sigma_S} \leq z\right) \approx \mathbf{\Phi}(z) - \varphi(z)\left(\frac{\kappa_{3S}}{6}H_2(z) + \frac{\kappa_{4S}}{24}H_3(z) + \frac{\kappa_{3S}^2}{72}H_5(z)\right),$$

with the standard normal density function $\varphi(x)$ and the Hermite polynomials $H_2(z) = z^2 - 1$, $H_3(z) = z^3 - 3z$ and $H_5(z) = z^5 - 10z^3 + 15z$.

**Gamma** The gamma approximation is the most famous approximation in this area (see e.g. Seal (1977)). Due to a translation $x_0$, three moments can be fitted

$$\begin{aligned} f_S(s) &\approx f_{gamma}(s) = \frac{\beta^\alpha (s-x_0)^{\alpha-1} e^{-\beta(s-x_0)}}{\Gamma(\alpha)} \\ \alpha &= (2/\kappa_{3S})^2, \quad \beta = 2/\left(\kappa_{3S}\sigma_S\right), \quad x_0 = \mu_S - 2\sigma_S/\kappa_{3S} \end{aligned}$$

**IG** This approximation is quite new (see Chaubey et al. (1998)), but surprisingly accurate.

$$\begin{aligned} f_S(s) &\approx f_{IG}(s) = \frac{\alpha}{\sqrt{2\pi\beta(s-x_0)^3}} \exp\left(-\frac{(\alpha-\beta(s-x_0))^2}{2\beta(s-x_0)}\right) \\ \alpha &= (3/\kappa_{3S})^2, \quad \beta = 3/\left(\kappa_{3S}\sigma_S\right), \quad x_0 = \mu_S - 3\sigma_S/\kappa_{3S} \end{aligned}$$

**Gamma-IG** The gamma-IG approximation is a combination of the gamma approximation and the IG approximation (see Chaubey et al. (1998)). Each of these approximations only uses the mean, standard deviation and skewness of $S$. Now a mixing parameter $w$ can be chosen such that the kurtosis of $S$ is fitted as well.

$$f_S(s) \approx f_{gamma-IG}(x) = w f_{gamma}(s) + (1-w) f_{IG}(s),$$

where $w = \left(\kappa_{4S} - \kappa_{4IG}\right) / \left(\kappa_{4gamma} - \kappa_{4IG}\right) = (\kappa_{4S} - 5\kappa_{3S}^2/3)/(-\kappa_{3S}^2/6)$.

## 2.2 Claim size distributions

There are quite a few claim size distributions available in literature. In principle, every density function on the positive axis that is unimodal, positively skewed and preferably closed under convolution can be used as a claim size distribution. In Kaas (2001) and Hogg and Klugman (1984) various claim size distributions and their properties are described. The most well-known ones are the gamma, lognormal and Pareto distribution. Some other possible claim size distributions include the Weibull, Burr and the inverse Gaussian distribution (IG). The latter is for instance used by Gendron and Crépeau (1989).

In this paper the most well-known claim size distributions mentioned above and the IG distribution are selected. Since several representations of these distributions exist, we shall first give the representations and the selected parameter values to be used in the present paper. We denote the random variable (r.v.) corresponding to the claim size distribution by $X$. Note that the mean does not really matter since all approximations are equivariant under scale transformations. For illustrative reasons, it is fixed at 100,000 for all claim size distributions. We also list the resulting skewness for each parameter combination; the resulting range is quite wide, which indicates that indeed a broad variety of claim size distributions is selected.

**Gamma**

$$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

The combinations $\left(\alpha, \beta^{-1}\right) = (400, 250)$, (100, 1000), (100/9, 9000), (4, 25000), (16/9, 56250), (1, 100000), (16/25, 156250) are used, corresponding to $\kappa_{3X} = 0.10$, 0.20, 0.60, 1.0, 1.5, 2.0, 2.5 respectively.

**IG**

$$f_X(x) = \frac{\alpha x^{-3/2}}{\sqrt{2\pi\beta}} \exp\left(-\frac{(\alpha - \beta x)^2}{2\beta x}\right)$$

The combinations $\left(\alpha, \beta^{-1}\right) = (400, 250)$, (100, 1000), (100/9, 9000), (4, 25000), (16/9, 56250), (1, 100000), (16/25, 156250) are used, corresponding to $\kappa_{3X} = 0.15$, 0.30, 0.90, 1.5, 2.25, 3.0, 3.75 respectively.

**Lognormal**

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\log(x) - \mu}{\sigma}\right)^2\right)$$

The combinations $(\mu, \sigma) = (11.44, 0.39)$, (11.27, 0.70), (11.07, 0.94), (10.88, 1.13), (10.71, 1.27) are used, corresponding to $\kappa_{3X} = 1.26$, 2.91, 5.33, 8.90, 14.0 respectively.

**Pareto**

$$f_X(x) = \frac{\phi\zeta^\phi}{(x + \zeta)^{\phi+1}}$$

The combinations $(\phi, \zeta \times 10^{-5}) = (21.5, 20.5)$, (8.20, 7.20), (5.56, 4.56), (4.43, 3.43) are used, corresponding to $\kappa_{3X} = 2.32$, 3.08, 4.10, 5.62 respectively. Note that a translated Pareto distribution is used that is defined for $x > 0$. Unfortunately, the very skewed Pareto distributions could not be selected since the kurtosis of $X$, and consequently the kurtosis of $S$, does not exist if $\phi$ is smaller than 4. This means that the fourth order approximations cannot be used.

## 2.3 Retention levels and criterion

The retention levels considered range from $\mu_S$ to $\mu_S + 3\sigma_S$ with steps of $\sigma_S/3$. In this way all situations of practical interest are amply covered. One of the new aspects in this paper is that we not only compare the approximations to each other, but that we also check whether or not the approximations satisfy a given criterion. Adding such an absolute yardstick clearly makes sense. The fact that an approximation comes out best in a given situation has a different meaning when all approximations perform well than when some (or most) are in fact too inaccurate to be used at all. Of course, the accuracy of the approximations of the stop-loss premium at retention $a$ depends on the value of $a$. Since the approximated distribution matches only the first three or four moments, the other moments will in general be different. This means as a rule that the farther we go into tail of the distribution (large $a$), the larger the relative difference between the approximated and the exact distribution will be. This also results in a larger relative error in the approximated stop-loss premium. Therefore, the criterion we choose is that the absolute value of the relative error has to be smaller than 2.5% at retention $\mu_S$, smaller than 30% at retention $\mu_S + 3\sigma_S$ and proportionally in between. The criterion is graphically illustrated in Figure 2 from section 3.1.

In the previous paragraph it was assumed that the exact stop-loss premium is tractable, but this is not always the case. In fact, reasonable results typically require that the individual claim size distributions have nice convolution properties. This is the case for the gamma and IG distributions. In the event of lognormal or Pareto distributed claim sizes, or always when the claims are dependent, it is nearly impossible to calculate the exact stop-loss premium. In the first case we have to resort to either simulation or to a numerical procedure such as described in Ströter (1985). In the second case simulation is the only option and hence this is the alternative approach we have used in both situations.

When the exact stop-loss premium cannot be calculated, matters become more complicated. Then the approximated stop-loss premiums have to be compared to simulated values. These simulated stop-loss premiums will be based on 100,000 simulations. Obviously, as the simulated stop-loss premiums are not exact, the criterion needs to be modified in this situation. Analysis of the simulated stop-loss premiums in the case of a gamma distributed claim size show that the maximum absolute value of the relative error of the simulated stop-loss premium is about 1.6% (28%) at retention $\mu_S$ ($+3\sigma_S$). The lognormal and Pareto distributions that we selected are a bit more extreme, so one might expect somewhat larger maximum errors there. In view of this, and for reasons of simplicity, the criterion is changed to 5% (60%) at retention $\mu_S$ ($+3\sigma_S$) for situations involving simulation. At first sight it may seem very liberal to allow errors as large as 60%. However, in Albers (1999) it was already demonstrated that ignoring small dependencies can cause errors of e.g. 500%. Hence in comparison, our criterion still makes sense.

# 3 Independence model

In the independence model we have a portfolio of $n$ contracts and assume that all the claims are independent. In this so-called individual risk model, the total claim amount $S$ is given by

$$S = \sum_{i=1}^{n} V_i X_i, \tag{1}$$

where $P(V_i = 1) = 1 - P(V_i = 0) = p$. In this model, the identically distributed $V_i$'s are Bernoulli variables indicating whether person $i$ has claimed at least once. The identically distributed $X_i$'s represent the total individual claim size on contract $i$ during the reference period, so one $X_i$ can consist of several claims. We assume that $V_1, ..., V_n, X_1, ..., X_n$ are independent. Throughout the paper, $n$ will be fixed at 10,000 and for this model $p$ is such that $p \times 10^3 \in \{1/2, 1, 2, 3, 4, 5, 6, 7, 8\}$. This is not a real limitation since the accuracy depends almost completely on the expected number of claims $np$, rather than on $n$ and $p$ individually. For ($n$,$p$)=(10000, 0.002) and ($n$,$p$)=(4000, 0.005) the graphs with the relative errors of the approximations are indistinguishable.

In order to approximate the stop-loss premiums, the mean, standard deviation, skewness and kurtosis of $S$ are required. These are obtained by utilizing $S \triangleq \sum_{j=1}^{N} X_j$, where the symbol $\triangleq$ stands for equality in distribution and the number of claims $N$ is a r.v. that is binomial($n$,$p$) distributed and independent of the $X_i$'s. The moments of $S$ can be expressed in the moments of $N$ and those of the $X_j$:

$$\begin{aligned}
\mu_{1S} &= \mu_{1N}\mu_{1X} \\
\mu_{2S} &= \mu_{2N}\mu_{1X}^2 + \mu_{1N}\mu_{2X} - \mu_{1N}\mu_{1X}^2 \\
\mu_{3S} &= \mu_{3N}\mu_{1X}^3 + 3\mu_{2N}\mu_{1X}\mu_{2X} - 3\mu_{2N}\mu_{1X}^3 + \mu_{1N}\mu_{3X} - 3\mu_{1N}\mu_{2X}\mu_{1X} + 2\mu_{1N}\mu_{1X}^3 \\
\mu_{4S} &= \mu_{4N}\mu_{1X}^4 + 6\mu_{3N}\mu_{1X}^2\mu_{2X} - 6\mu_{3N}\mu_{1X}^4 + 3\mu_{2N}\mu_{2X}^2 - 18\mu_{2N}\mu_{1X}^2\mu_{2X} + 4\mu_{2N}\mu_{1X}\mu_{3X} \\
&\quad + 11\mu_{2N}\mu_{1X}^4 + \mu_{1N}\mu_{4X} - 4\mu_{1N}\mu_{3X}\mu_{1X} + 12\mu_{1N}\mu_{2X}\mu_{1X}^2 - 3\mu_{1N}\mu_{2X}^2 - 6\mu_{1N}\mu_{1X}^4,
\end{aligned}$$

where the notation $\mu_{iY} = EY^i$ is used. From these moments, the mean, standard deviation, skewness and kurtosis are easily obtained. An alternative way to obtain the moments of $S$ is by using similar relations between the cumulants of $S$ on the one hand and those of $N$ and $X_j$ on the other hand (see Panjer and Willmot (1992)).

For all combinations of claim size distributions from Section 2 and above stated values of $p$, we will compare the approximated stop-loss premiums to the exact stop-loss premiums, whenever the latter can be calculated. If the claim distribution is the lognormal or the Pareto distribution or if the claims are not independent (Section 4), the exact stop-loss premium cannot be calculated. In this case, we will compare the approximated stop-loss premiums to simulated values. This comparison means that the relative error is calculated at several retention levels.

## 3.1 Typical examples

The following typical examples illustrate the relative differences in accuracy between the various approximations. In Figure 1, the claim size distribution is not very skewed and the expected number of claims is 50. The resulting total claim amount has a small skewness and kurtosis. In this case, clearly the gamma-IG approximation is the most accurate one, followed by the Edgeworth approximation. However, even the worst approximation is still very accurate, even at a large retention.

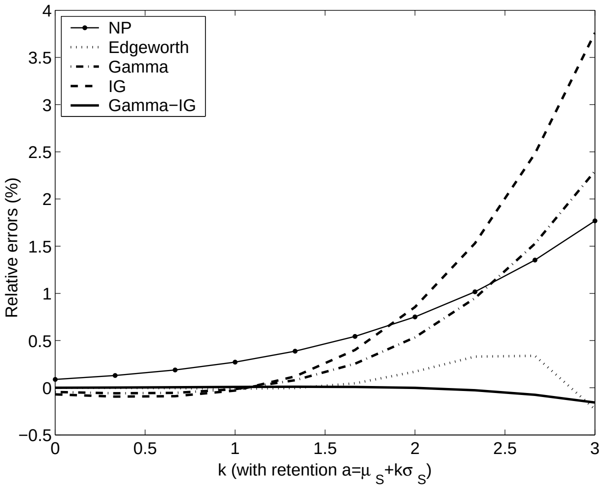

Figure 1. The relative error of the various approximations when X∼IG(4, 1/25000) and $p$=0.005. This leads to $\kappa_{3X}$=1.5, $\kappa_{3S}$=0.20 and $\kappa_{4S}$=0.047.

In Figure 2, the skewness of the claim size distribution is moderately large. The errors of especially the Edgeworth and gamma-IG approximation are quite large, even at a small retention. Neither of these satisfies the criterion which runs from ±2.5% to at $k = 0$ to ±30% at $k = 3$. In this case, the IG approximation is the best, followed by the gamma approximation.

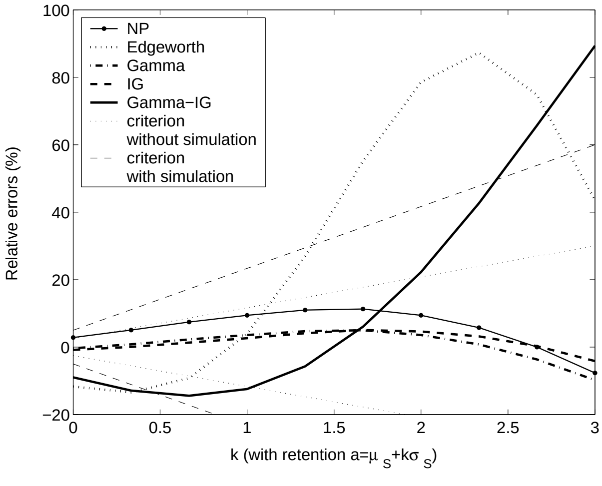

Figure 2. The relative error of the various approximations when X∼Pareto(4.43, 3.43×$10^5$) and $p$=0.002. This leads to $\kappa_{3X}$=5.62, $\kappa_{3S}$=0.96 and $\kappa_{4S}$=4.06.

## 3.2 Results

For each combination of claim size distribution and value of $p$ the relative error of all approximations at the different retentions have been plotted, leading to a multitude of graphs like Figures 1 and 2 from the previous subsection. These plots are used to analyze which factors influence the accuracy of the approximations most and which approximation is best. Furthermore we have checked in each case whether the approximations satisfy the criterion.

The magnitude of the relative errors can be characterized mainly by the kurtosis $\kappa_{4S}$ of the total claim amount. The differences between the approximations depend mainly on the skewness $\kappa_{3X}$ of the claim size distribution. For claim size distributions with a small skewness, the gamma-IG approximation is the most accurate approximation, followed by the Edgeworth approximation. If the skewness is larger, these approximations become very inaccurate. This is due to the fact that generally the kurtosis of the claim size distribution becomes very large as well, and so will the kurtosis of the total claim amount. The correction for the kurtosis made by the fourth order approximations dominates the approximations. In this case, the IG approximation is the most accurate approximation, followed by the gamma approximation.

When the claim size is gamma or IG distributed, all approximations satisfy the criterion in most cases. For the extreme lognormal and Pareto distributed claim sizes, the IG and gamma approximation meet the criterion in most cases while the other approximations do not. The analysis of the differences between the approximations and the factors that influence the accuracy leads to the following rule of thumb:

$$\begin{array}{ccccc} 0 \le \kappa_{3X} \le 5 & \wedge & 0 \le \kappa_{4S} \le 1.5 & \Rightarrow & \text{Gamma-IG approximation} \\ 5 < \kappa_{3X} < 15 & \vee & 1.5 < \kappa_{4S} < 50 & \Rightarrow & \text{IG approximation} \end{array} \tag{2}$$

This rule provides the best approximation in almost all cases. Otherwise, it leads to at least an approximation that satisfies the criterion. The only two exceptions are extreme cases: the most skewed lognormal claim size distribution and $p$=0.0005 or $p$=0.001. In these cases the kurtosis of the total claim amount $\kappa_{4S}$ was 125 and 62.5 respectively. Generally, this rule of thumb (2) can be safely extended to situations where $np$ is larger than 80. As the expected number of claims increases, the distribution of $S$ will converge to a normal distribution which can be accurately approximated by the gamma-IG approximation. An extension to $np < 5$ on the other hand is rather tricky and is not recommended.

# 4 Dependence model

In the dependence model there are still $n$ contracts in the portfolio, but now we introduce some dependence in the sense that whole groups can submit claims. The portfolio is divided into $h = n/g$ groups, each consisting of $g$ contracts. Some of these groups are collectively victim of a common risk. If so, it is assumed that all group members submit at least one claim due to this special cause. The other people submit at least one claim with probability $p$. This leads to:

$$S = \sum_{i=1}^{gW} X_{1i} + \sum_{j=1}^{n-gW} V_j X_{2j}, \tag{3}$$

where $P(V_j = 1) = 1 - P(V_j = 1) = p$. Hence in this model there are $W$ groups victim of a common risk. This number is assumed to be binomial($h$,$q$) distributed. Note that $q = 0$ reduces the model to the independence model (1). The $X_{1i}$'s represent the total individual claim sizes of the people who are victim of a common risk. The $X_{2j}$'s represent the total individual claim sizes of the other people. Just as in the individual risk model (1), $V_j$ is a Bernoulli variable indicating if person $j$ has submitted at least one claim. People who are victim of a common risk also have their usual claim possibility as everyone and hence $X_{1i}$ consists in fact of two parts: the claim due to the common risk, which we call the dependent claim, and the usual claim, to which we refer as the independent claim. Therefore, it makes sense to choose $X_{1i}$ stochastically larger than $X_{2j}$.

The $X_{1i}$'s are identically distributed, and the same holds for both the $X_{2j}$'s and the $V_j$'s as well. Moreover the $X_{1i}$'s, $X_{2j}$'s and $V_j$'s are independent. In our calculations the $X_{1i}$'s and the $X_{2j}$'s will come from the same family of distributions and have the same skewness. But this still leaves the possibility of different expectations. To avoid having to inspect hundreds of situations, we choose to only consider lognormal distributed claim sizes and $p = 0.008$. This is a reasonable restriction, since it was concluded in the previous section that the skewness of the claim size is important, rather than the family of the claim size. For values of $p$ even smaller than 0.008, the total claim amount will be dominated by the dependent collective claims and it cannot reasonably be expected that the approximations are accurate.

In comparison to the dependence model, we now have three extra parameters which determine the level of dependence. These are the group size $g$, the probability that a group is victim of a common risk $q$ and the ratio $r = EX_{1i}/EX_{2j}$. In this paper we choose the following parameter values: $q \times 10^5 \in \{8, 24, 40\}$, $g \in \{5, 10, 20\}$, $r \in \{1, 2, 3\}$.

The values for $q$ are such that the expected number of dependent claims (thus the $X_{2j}$'s) is about 1%, 3% and 5% of the expected number of independent claims. The group size ranges from about 1/16 to 1/4 times the expected number of claims. Hence, with these parameter values the total claim amount will definitely not be dominated by the collective claims, as these only contribute a fraction of the total damage. Nevertheless, in the next section it will be shown that the choice of parameter values above leads to stop-loss premiums that range from just over 1 to 500 times the stop-loss premiums for the model with independent claims. Hence even considerably stronger than in the simple normal case from Albers (1999) the message is that ignoring even small dependencies can lead to massive relative errors in quantities such as the stop-loss premiums, which deal with the tail area of the distribution. In Section 5 we will devote specific attention to this obviously important conclusion.

Again it is useful to briefly describe how the moments of $S$ can be obtained. The main trick is to condition on $W$. For a fixed $W$, the two sums ($Y_1 = \sum_{i=1}^{gW} X_{1i} = \sum_{i=1}^{gW} U_i X_{1i}$ with $P(U_i = 1) = 1$ and $Y_2 = \sum_{j=1}^{n-gW} V_j X_{2j}$) in (3) are in principle similar to the sum in (1). Hence the moments of $Y_1$ and $Y_2$ for fixed $W$ are easily obtained. The final step is to obtain the unconditional moments of $S = Y_1 + Y_2$, the first of which is given by $E(S) = E(E(Y_1|\,W) + E(Y_2|\,W))$. The others are obtained by expanding $S^k = (Y_1 + Y_2)^k$.

## 4.1 Typical examples

The following typical graphs illustrate the effect of the dependency. In Figure 3, the skewness of the claim size distribution is very large. The dependence however is minimal. Note the similarity of Figure 2 of section 3.1 and Figure 3. Again, the IG and the gamma approximation are best.

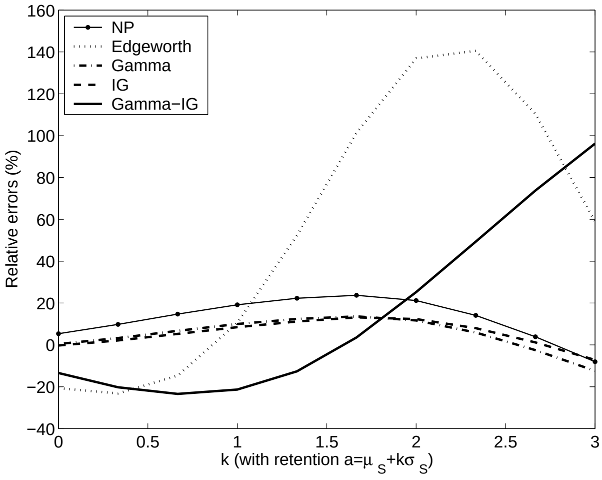

Figure 3. The relative error of the various approximations when X∼lognormal(10.71, 1.27), $p$=0.008 and ($g$,$q$,$r$)=(5, 0.00040, 1). This leads to $\kappa_{3X}$=14.0, $\kappa_{3S}$=1.19 and $\kappa_{4S}$=6.99.

Figure 4 clearly illustrates the effect of the dependence. Here, the dependence parameters are rather large. The wave shape of the approximations is caused by the bimodal character of the distribution of the total claim amount. In this case, none of the approximations meets the criterion.

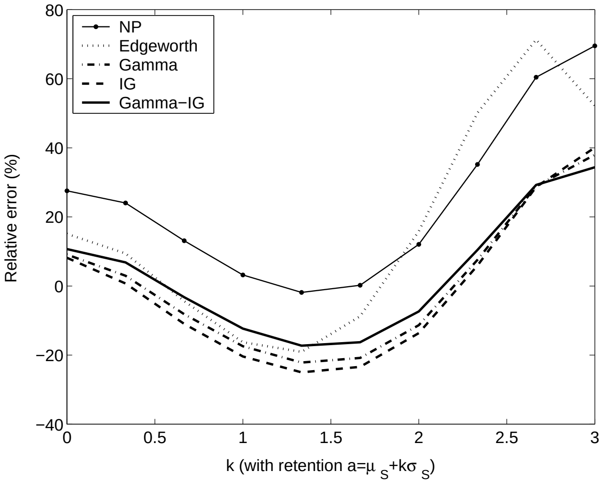

Figure 4. The relative error of the various approximations when X∼lognormal(11.44, 0.39), $p$=0.008 and ($g$,$q$,$r$)=(20, 0.00024, 3). This leads to $\kappa_{3X}$=1.26, $\kappa_{3S}$=2.20 and $\kappa_{4S}$=5.84.

### 4.2 Results

If the dependence is minimal, the results are quite similar to those from the independence model. The order of magnitude of the error is mainly determined by the skewness and the kurtosis of $S$. The differences are mainly determined by the skewness of the claim size distribution. For small skewnesses the gamma-IG approximation is the most accurate one, for large skewness the IG approximation is the most accurate one.

If there is a bit more dependence, the approximations become more and more inaccurate. This is caused by the bimodal character of the density function of the total claim amount. All the approximations are essentially based on unimodal densities. Consequently there are quite a few cases where the approximations do not satisfy the criterion, especially when there is a bit more dependence. In these cases the stop-loss premium has to be simulated. Nevertheless, if the following modified rule of thumb is used, the resulting approximation does meet the modified criterion from Section 2.

$$\begin{array}{rcrcc} g/(np) \geq \frac{1}{4} & \wedge & r \geq 2 & \Rightarrow & \text{simulation} \\ 0 \leq \kappa_{3X} \leq 5 & \wedge & 0 \leq \kappa_{4S} \leq 1.5 & \Rightarrow & \text{Gamma-IG approximation} \\ 5 < \kappa_{3X} < 15 & \vee & 1.5 < \kappa_{4S} < 50 & \Rightarrow & \text{IG approximation} \end{array} \tag{4}$$

Again, one has to be careful before extending the rule of thumb outside the parameter range used in the present article. Especially the extensions in the extreme direction are not recommended, meaning the direction of more dependence (e.g. through allowing larger groups), less expected claims or more skewed claim size distributions.

## 5 Comparison of the models

In the introduction it was mentioned that most insurance companies ignore the dependence between claims. It is interesting to see what the effect is of the dependence on the stop-loss premiums. If there exists a small probability of a huge total claim amount, this might have major consequences for the stop-loss premium. Therefore we compare the approximated stop-loss premiums of the model with dependent claims of Section 4 to the ones of the individual risk model of Section 3 with $p = 0.008$ and lognormal claims size distributions. For the sake of a fair comparison, the claim size distributions and the value of $p$ are slightly adjusted for the model with dependent claims, such that the expected number of claims and the expected total claim amount is equal for both models. Furthermore, both these stop-loss premiums were calculated at the retention levels belonging to the individual risk model.

### 5.1 Results

In the following figures the relative difference between the stop-loss premiums of both models are given. In Figure 5 the dependence is very weak and the difference thus small. In Figure 6 the dependence is ´intermediate´ and in Figure 7 there is a ´strong´ dependence. Here, the quotation marks are used since in all cases the expected number of dependent claims is at most 5% of the expected number of independent claims and the level of dependence is thus rather small in all considered cases.

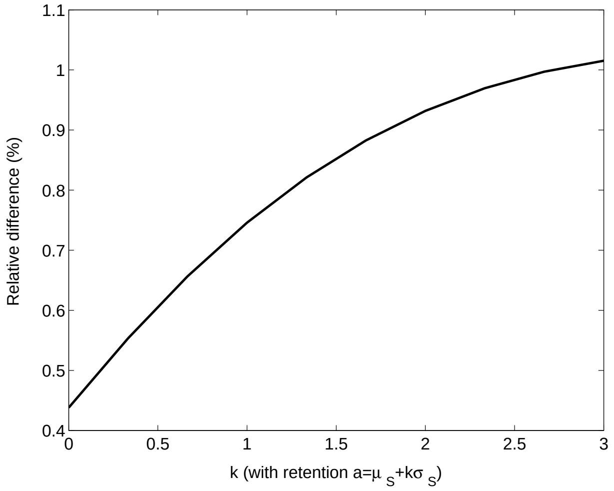

Figure 5. Relative difference between the stop-loss premium in the model with dependent claims and the model with independent claims. The dependence parameters are $(g, q, r)$ = (5, 0.00008, 1) while X∼lognormal(10.71, 1.27) and $p$=0.008.

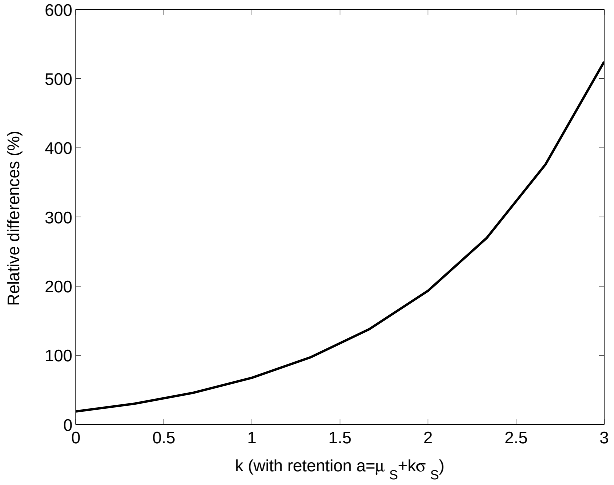

Figure 6. Relative difference between the stop-loss premium in the model with dependent claims and the model with independent claims. The dependence parameters are $(g, q, r)$ = (10, 0.00024, 2) while X∼lognormal(11.07, 0.94) and $p$=0.008.

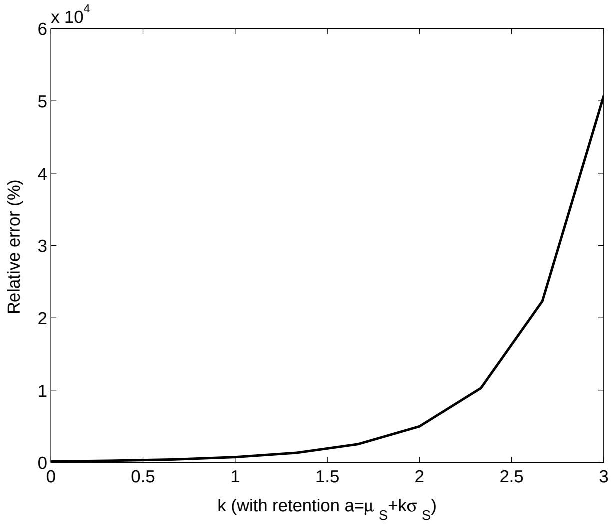

Figure 7. Relative difference between the stop-loss premium in the model with dependent claims and the model with independent claims. The dependence parameters are $(g, q, r)$ = (20, 0.00040, 3) while X∼lognormal(11.44, 0.39) and $p$=0.008.

In Figure 7 we see that the stop-loss premium at retention $\mu_S + 3\sigma_S$ is about 50,000% larger than in the individual risk model. This means that the stop-loss premium of the model with dependent claims exceeds that of the model with independent claims by a factor 500, which is remarkable considering the small level of dependence.

# 6 Conclusion

The accuracy of the approximations depends most of all on the skewness $\kappa_{3X}$ of the claim size distribution and the kurtosis $\kappa_{4S}$ of the total claim amount. The approximations are quite accurate, even when some dependence is introduced. If there is a strong dependency between the claims, simulation of the stop-loss premiums is recommended. When the rule of thumb (4) is used, the resulting approximation satisfies the criterion. Even for a rather small level of dependence as an expected number of dependent claims being 5% of the expected number of independent claims, the stop-loss premiums are substantially larger. Of course, this should not be ignored.

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