Department of Applied Mathematics
Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands

Phone: +31-53-4893400
Fax: +31-53-4893114

Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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W. Albers and W.C.M. Kallenberg

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Alternative Shewhart-type charts for grouped observations

Willem Albers and Wilbert C.M. Kallenberg
Department of Applied Mathematics
Faculty of Electrical Engineering, Mathematics and Computer Science
University of Twente
P.O. Box 217, 7500 AE Enschede
The Netherlands

Abstract Classical Shewhart control charts based on groups of observations use the average as statistic to decide whether a signal should be produced or not. Here several other statistics are investigated as well. Moreover, individual charts are compared with group charts. The chart using the minimum of each group of observations is a promising new proposal. For this chart (nonparametric) estimation of an extreme quantile is reduced to estimation of an ordinary quantile, which is far easier to perform accurately. The behavior of the minimum-chart is competitive to that of the average-chart when distributions are known.

Keyword and phrases: Statistical Process Control, Phase II control limits, order statistics.

2000 Mathematics Subject Classification: 62 P 30, 62 C 05.

1 Introduction

Consider the case where the mean of a production process is monitored using a Shewhart chart based on (groups of) incoming measurements. An upper limit and a lower limit are set and as soon as either of these is exceeded for a newly arriving (group of) measurement(s), an Out-of-Control (OoC) signal occurs. While the process is in fact In-Control (IC), the resulting false alarm rate (FAR) should equal some very small quantity p, typically of the order 0.001. Almost always, the underlying distribution generating the measurements is unknown and a sample of so-called Phase I observations is required to estimate the limits of the chart before the actual control can begin.

Standard practice assumes normality, which reduces the problem to estimating the normal mean and variance involved. However, even in this relatively simple setup, it is by now rather well-known that a very large sample of Phase I observations is required before the estimation errors become sufficiently small to be safely ignored. See e.g. Ghosh et al. (1981), Quesenberry (1993), Roes (1995), Chen (1997), Woodall and Montgomery (1999) (p.379) and Chakraborti (2000). Therefore, Albers and Kallenberg (2003, 2004a,b) (to be denoted for short as AK (2003, 2004a, b) in the sequel) have demonstrated how this can be solved by using relatively simple corrections.

But the normality assumption itself is often questionable as well, as was pointed out before by several authors, see e.g. Chan et al. (1988), Pappanastos and Adams (1996). Without normality, the resulting estimation problem is essentially more complicated and so far results are mainly restricted to the case of individual measurements, i.e. where the observations arrive one at a time and the decision to produce an alarm or not is taken immediately. In fact, for this situation Albers, Kallenberg and Nurdiati (2002, 2004) (AKN (2002, 2004) for short) have extended the usual normal charts to parametric ones. Essentially, in addition to mean and variance, a shape parameter is estimated there as well. As a further alternative, nonparametric charts are considered in AK (2004c). (For some closely related charts see Willemain and Runger (1996) and Ion et al. (2000); for a recent overview of nonparametric charts in general, see e.g. Chakraborti et al. (2001).) Finally, AKN (2003) present a data driven procedure to select the best solution in a given case from among the normal, parametric and nonparametric choices offered, see also AK (2004d). Attractive aspects of this latter approach are that one sticks with the (corrected!) normal chart as long as the data permit; if the departure from normality apparently is too strong, the parametric alternative kicks in; only when the data indicate that the distribution differs so much from normality that even the parametric model is no longer adequate, one has to resort to the nonparametric chart.

The data driven control chart seems to be the best thing to do when the chart is based on individual measurements. Nevertheless, it is not completely satisfying, since in particular when the nonparametric chart comes in, the required number of Phase I observations is pretty large, say at least 500. The reason for it is the estimation of an extreme quantile, like the upper 0.001-quantile. It is clear that for the fully nonparametric chart even much larger sample sizes than 500 are needed before a reasonable solution exists. However, with 500 observations or more a so called modified nonparametric chart works well for quite a lot of distributions also covering distributions far away from normality, see AKN (2003) and AK (2004d).

To get suitable nonparametric charts for less than 500 Phase I distributions it seems necessary to use groups of observations. The advantage is that we do not have to decide immediately when a Phase II observation arrives, but that we may postpone until some more observations appear. Clearly, $2, 3$ or 4 observations tell us more than only 1. Taking for instance the minimum of 3 observations as yardstick, we have to deal with its upper 0.003-quantile to get the average run length (ARL) equal to 1000 and that corresponds to the 0.144-quantile of the original distribution. That implies that estimating of the extreme 0.001-quantile is replaced by estimating the 0.144-quantile, which can be done very well with 50 Phase I observations, say.

The disadvantage of working with a group of $m > 1$ Phase II observations is that we cannot stop at an earlier time than after m steps. Moreover, the next possible stop is at time $2m$ etc. We have jumps of size m in stopping the process. Moreover, when the process gets OoC we may encounter a group of observations with a mix of IC- and OoC-observations. Therefore, we should take m rather small: $2, 3, 4$ or 5. When restricting to those m's, it is clear that in case of IC, where with $FAR = 0.001$ the ARL equals 1000, a difference in RL of m observations is completely negligible and hence the discrete character of RL and the possible mix of IC- and OoC-observations does not matter. The OoC situation is slightly different, but we concentrate on changes in the process with ARL between 10 and 200. Hence, also here we do not bother too much about differences in detecting of a few more observations.

In the individual case, it is trivially clear that a signal will arise if either the new observation is too large or too small. But in the grouped case, first the question has to be dealt with which statistic based on the m observations, should actually be used. Under normality, the answer is straightforward: the sample mean is optimal and easy to work with. In fact, in a few simple steps the case $m > 1$ is reduced to the case $m = 1$. Beyond the normal model, the picture is quite different, however. The sample mean is not necessarily optimal, and it also is not particularly easy to deal with (remember that $m \leq 5$, so the central limit theorem is not of much use here, especially not as the interest is focused on the tails of the distribution).

Hence the subject of the present paper will be the study of a variety of possible statistics for use in grouped control charts with main attention on the nonparametric chart, but the normal chart will be discussed as well. One aspect will obviously be how efficient a particular choice is: given a certain FAR, how large is the probability of detection during OoC offered by the choice made? Another criterion will be its ease of application. Moreover, note that in fact two types of comparisons play a role. In the first place, for each fixed value of m, various statistics can be compared. But each given type of statistic can also be compared for varying m. Even the normal case is not quite trivial in this respect and still leads to some interesting insights. The point is of course that we are not dealing with a single given OoC-situation, implying that the optimal choice of m will vary according to the alternative considered.

It turns out that answering the various questions raised in the previous paragraph already poses quite a task in itself. Hence it seems wise to make a similar division as in the individual case. The first step thus is to figure out these answers for a known (but not necessarily normal) underlying distribution. Once a more or less clear picture has been obtained about which statistics have which properties under which conditions, the second step can be taken. This will entail the estimation of the parameters and/or distributions involved, the study of the estimation effects incurred and the derivation of possible corrections for errors which are considered to be intolerably large. In the present paper we shall address the first step, and thus work under the assumption of a known underlying distribution. But estimation is nevertheless present in the background, since the statistics involved should be related to corresponding nonparametric control charts and the possibilities for estimation and its consequences should be taken into account. The latter concerns only a first impression. The full second step, concerning the estimation aspects, will be dealt with in a forthcoming paper.

The paper is structured as follows. In section 2 the general approach is given. Here we introduce the setup, the notation involved and we present the various statistics for use in grouped observations and the general form of their control limits. Among these statistics is the minimum of the m observations. As far as we know the chart based on this choice is new. Next, in section 3, we compare the several methods and the size m of the groups of observations. We start with the normal distribution followed by some symmetric distributions (random normal mixture) which are not too far from normality and end with a skew distribution, representing a distribution farther away from normality. Section 4 gives a first sketch of the nonparametric estimation problem with emphasis on the consequences for choosing a certain statistic. It turns out in Sections 3 and 4 that the new chart based on the minimum is very promising. A brief summary of the conclusions is given in section 5.

2 General approach

The basic set-up is as follows. Consider random variables $X_1, ..., X_n, X_{n+1}, ..., X_{n+N}, X_{n+N+1}, ...$, where $X_1, ..., X_n$ is a sample of so-called Phase I observations and $X_{n+1}, ..., X_{n+N}, X_{n+N+1}, ...$ are the Phase II observations, the first N being IC and the last part OoC. The number N is unknown. It is assumed that the Phase I observations are IC and we use them for estimating unknown quantities in the basic control limits and in addition for making necessary corrections after plugging in the estimators. In this paper we do not concentrate on the estimation part (except for motivational purposes) and therefore we deal with the Phase II observations.

The prescribed FAR will be denoted by p. For ease of presentation we concentrate on the one-sided case where only an upper limit is needed. For any distribution function (df) H we will write $\overline{H} = 1 - H$, and H^{-1} and $\overline{H}^{-1}$ for the respective inverse functions. (Observe that the inverse is defined unambiguously for continuous and increasing H; for the remaining cases a choice has to be specified, but we will tacitly assume in this paper that all df's are continuous and increasing.) Let $X_{n+1}, ..., X_{n+N}$ have df F and let $X_{n+N+1}, ...$ be distributed as $X + d$ with X having df F and $d = d(F)$, such that

$$P\left(X + d > \overline{F}^{-1}(p)\right) \in \left(\frac{1}{200}, \frac{1}{10}\right). \tag{1}$$

This corresponds with an ARL between 10 and 200, when using the individual chart. Such changes in the process are the most interesting ones when dealing with Shewhart charts: for smaller changes the Shewhart chart is less natural (cf. discussions about the relative merits of Shewhart and CUSUM charts), larger changes are detected anyhow in a few steps. As usual with Shewhart charts the Phase II observations are not used for estimation, even if they have passed the chart without producing an alarm. As indicated we do not bother about small differences in RL and therefore we may assume also for the group of observations that they are either all IC or all OoC.

Hence, from now on our set-up is more simple: we have a group of observations $Y_1, ..., Y_m$ (with $m = 1, ..., 5$, thus including the individual chart as well), which is either IC, that is they are distributed as X, or they are OoC and are distributed as $X + d$ with d in principle given by (1). A chart is defined by a statistic $w(Y_1, ..., Y_m)$ and an upper limit $UL(w, m)$ and an alarm is produced when

$$w(Y_1, ..., Y_m) > UL(w, m).$$

To compare the charts for different values of m in a fair way we match the ARL's under IC. Hence, writing $F_{w,m}$ for the df of $w(Y_1, ..., Y_m)$ in the IC case, we have

$$UL(w, m) = \overline{F}_{w,m}^{-1}(mp). \tag{2}$$

The performance of several statistics $w(Y_1, ..., Y_m)$ (and several values of m) will be investigated by their ARL under OoC: the smaller the ARL, the better the chart. (But note that small differences are irrelevant.) In the following subsections various choices of the statistics $w(Y_1, ..., Y_m)$ are presented.

2.1 AVE

To begin with we consider the obvious choice (at least under normality), which is the average-chart (AVE), based on

$$w(Y_1, ..., Y_m) = m^{1/2}\overline{Y}.$$

When normality holds this clearly is the optimal choice, but also in a nonparametric context it is of interest. Let us briefly digress into the estimation case. Using averages while F is unknown brings us into the area of normal permutation tests. Some efforts of this type were mentioned in Chakraborti et al. (2001). For example, Alloway and Raghavachari (1991) use a procedure based on the Hodges-Lehmann estimator and work with Walsh averages. However, as pointed out by Chakraborti et al. (2001) and by Pappanastos and Adams (1996), the resulting charts are in fact not truly nonparametric or distributionfree. Their actual in-control run length distribution involved does depend on the underlying distribution of the observations.

2.2 UNI

A second statistic is defined by

$$w\left(Y_1, ..., Y_m\right) = \sum_{i=1}^{m} F(Y_i). \tag{3}$$

As $F(Y_i)$ is uniformly distributed under IC, we call this UNI. Replacing F in $F(Y_i)$ by F_n, the empirical distribution function of the Phase I observations $X_1, ..., X_n$, we get that $nF_n(Y_i) = R(Y_i) - 1$, where $R(Y_i)$ is the rank of Y_i among $X_1, ..., X_n, Y_i$. Therefore, UNI is the limiting form of the sum of ranks based statistic. Hence, the statistic in (3) produces a Wilcoxon-type of approach in the estimated version, and as such offers a likely and possibly attractive alternative to the standard parametric approach AVE. In the review on nonparametric charts by Chakraborti et al. (2001), rank based charts of this nature by e.g. Bakir and Reynolds (1979) and Hackl and Ledolter (1991, 1992) are mentioned. However, here we shall refrain from going into the estimation aspects and concentrate on the performance under known F. Hence, ranks will remain in the background and uniform-charts are the ones we focus on.

Let $U_1, ..., U_m$ be a sample from the uniform distribution on $(0, 1)$. Then $\sum_{i=1}^{m} F(Y_i)$ has under IC the same df as $\sum_{i=1}^{m} U_i$. To determine the upper limit $UL(w, m)$ for this case, we begin by observing that for $c \le 1$ we simply have

$$P\left(\sum_{i=1}^{m} U_i > m - c\right) = P\left(\sum_{i=1}^{m} U_i < c\right) = \frac{c^m}{m!}.$$

Hence, $\overline{F}_{w,m}^{-1}(mp)$ is obtained by taking $c = \{m!(mp)\}^{1/m}$, which result will indeed be ≤ 1 for $m \le 5$ and $p = 0.001$. (Of course, for larger c the result can also be readily obtained, but for ease of presentation we concentrate on this most simple case.) Therefore,

$$UL(UNI, m) = m - \{m!(mp)\}^{1/m} \text{ for } m \le 5 \text{ and } p = 0.001. \tag{4}$$

Note that for $m = 1$ this leads to $F(Y_1) > 1 - p$, which indeed agrees with the requirement $Y_1 > \overline{F}^{-1}(p)$, used in the individual chart.

2.3 MAX, MIN and MIX

Instead of ranks we may also consider order statistics. Under OoC we have a shift to the right and hence in general larger values than under IC. Hence, the first idea might be to take MAX, the largest of $Y_1, ..., Y_m$, that is

$$w\left(Y_1, ..., Y_m\right) = \max\left(Y_1, ..., Y_m\right).$$

But this first idea is not so brilliant. In fact, MAX is very close to using m times the individual chart. For instance, when $m = 2$, the difference between $P\left(\max\left(Y_1, Y_2\right) > y\right)$ and $P\left(Y_1 > y\right) + P\left(Y_2 > y\right)$ is only $P\left(Y_1 > y, Y_2 > y\right)$ which is very small, since already $P\left(Y_1 > y\right)$ is small. The advantage of taking a group is hardly used: when we have under IC a group with a large value, as a rule it will be the only one in the group and hence it is just like taking m individual charts.

Under IC we have

$$\overline{F}_{MAX,m}(y) = 1 - P\left(\max\left(Y_1, ..., Y_m\right) \leq y\right) = 1 - \{F(y)\}^m$$

and hence we obtain as upper limit, cf. (2),

$$UL(MAX, m) = \overline{F}_{MAX,m}^{-1}(mp) = \overline{F}^{-1}(1 - \{1 - mp\}^{1/m}). \tag{5}$$

Indeed, for small p, it holds that $1 - \{1 - mp\}^{1/m} \approx p$, which shows that approximately MAX does nothing but finish the series of m observations in which the individual chart has given a signal.

Taking MIN, the smallest of $Y_1, ..., Y_m$, that is

$$w\left(Y_1, ..., Y_m\right) = \min\left(Y_1, ..., Y_m\right)$$

looks more promising. On the one hand when under OoC a shift occurs, we have this shift also in the next observations. When using MIN we take advantage of this effect that in a group the observations intensify each other. That is, already if m observations are pretty large and not necessarily extremely large, this is enough evidence to give an alarm. Here really the group is used. Because under IC

$$\overline{F}_{MIN,m}(y) = P\left(\min\left(Y_1, ..., Y_m\right) > y\right) = \left\{\overline{F}(y)\right\}^m$$

we get as upper limit, cf. (2),

$$UL(MIN, m) = \overline{F}_{MIN,m}^{-1}(mp) = \overline{F}^{-1}(\{mp\}^{1/m}). \tag{6}$$

As concerns estimation, (6) immediately shows that MIN reduces estimation of the far tail to estimation of the ordinary tail, see also the Introduction, where we encountered $0.144 = (0.003)^{\frac{1}{3}}$.

Apart from MAX and MIN we consider a mixture of the two, called MIX, where an alarm is produced if for some probability s and some $0 \leq \gamma \leq (1 - mp)^{1/m}$

$$\min\left(Y_1, ..., Y_m\right) > \overline{F}^{-1}(s) \text{ and } \max\left(Y_1, ..., Y_m\right) > \overline{F}^{-1}\left((1 - \gamma)\, s\right). \tag{7}$$

It is immediate to see that during IC the event in (7) has probability $s^m\left(1 - \gamma^m\right)$. If as before, the comparison is made fair again by setting this equal to mp, it follows that

$$s = \left(\frac{mp}{1 - \gamma^m}\right)^{1/m} \text{ or } \gamma = \left(1 - \frac{mp}{s^m}\right)^{1/m}. \tag{8}$$

Letting γ increase from 0 to $(1 - mp)^{1/m}$ it easily follows from (8) that we go from MIN to MAX, cf. also (6) and (5).

The FAR of MIX during OoC is given by

$$P\left(\min\left(Y_1^{(0)}, ..., Y_m^{(0)}\right) + d > \overline{F}^{-1}(s), \max\left(Y_1^{(0)}, ..., Y_m^{(0)}\right) + d > \overline{F}^{-1}\left((1 - \gamma)\, s\right)\right),$$

where $Y_1^{(0)}, ..., Y_m^{(0)}$ still have df F. Direct calculation gives

$$\begin{aligned} &P\left(\min\left(Y_1^{(0)}, ..., Y_m^{(0)}\right) + d > \overline{F}^{-1}(s), \max\left(Y_1^{(0)}, ..., Y_m^{(0)}\right) + d > \overline{F}^{-1}\left((1 - \gamma)\, s\right)\right) \\ &= \left\{\overline{F}\left(\overline{F}^{-1}(s) - d\right)\right\}^m - \left\{\overline{F}\left(\overline{F}^{-1}(s) - d\right) - \overline{F}\left(\overline{F}^{-1}((1 - \gamma)\, s) - d\right)\right\}^m \end{aligned}$$

and hence the ARL of MIX during OoC gives

$$ARL(MIX, m, d) = \frac{m}{\left\{\overline{F}\left(\overline{F}^{-1}(s) - d\right)\right\}^m - \left\{\overline{F}\left(\overline{F}^{-1}(s) - d\right) - \overline{F}\left(\overline{F}^{-1}((1-\gamma)\,s) - d\right)\right\}^m}$$

with s or γ given by (8). In particular we get for MIN

$$ARL(MIN, m, d) = \frac{m}{\left\{\overline{F}\left(\overline{F}^{-1}((mp)^{1/m}) - d\right)\right\}^m}$$

and for MAX

$$ARL(MAX, m, d) = \frac{m}{1 - \left\{F\left(\overline{F}^{-1}(1 - (1-mp)^{1/m}) - d\right)\right\}^m}.$$

As far as we know the MIX control charts are new and hence in particular the control chart based on the minimum of a group of observations is new.

3 Comparison

In this section we compare the various methods AVE, UNI, MAX, MIN, MIX, with for each of them the values $m = 1, 2, 3, 4, 5$. For all these charts, the case $m = 1$ produces the same result: the individual chart (IND), giving a signal when $Y_1 > \overline{F}^{-1}(p)$. To fix ideas, for the FAR we choose $p = 0.001$, unless stated otherwise. The shifts d are in principle according to (1). The IC behavior is put on a par by matching the ARL's, see (2). Therefore, the comparison concerns the ARL's during OoC: the smaller they are, the better the method with the restriction that small differences are not taken into account. We present (mostly) the difference in ARL between a certain method at some m and the individual chart IND ($m = 1$). In case of MIX we take $\gamma = 1 - (4m)^{-1}$, which turns out to be the nearly best value under normality. With regard to the df's we consider the normal distribution, some members of the family of random normal mixtures and the Gamma(4, 1)-distribution. The random mixtures under consideration are still symmetric and most of them not that far from normality. The Gamma(4, 1)-distribution is skew and farther away from the normal distribution. For all the distributions we take without loss of generality the expectation of the observations under IC equal to 0 and the variance equal to 1.

To quantify the "distance" of the distributions considered here from normality in the context of control charts we calculate the so called model error (ME) for $p = 0.001$. This ME is the probability of a false alarm when acting as if still normality holds, that is applying the (individual) normal chart under the given df. In formula $ME = \overline{F}^{-1}(u_p) - p = \overline{F}^{-1}(u_{0.001}) - 0.001$, where u_p is the upper p-quantile of the standard normal distribution. Note that when normality holds, indeed we get $ME = 0$. For the random normal mixtures of Section 3.2 we get $0.0041, 0.0083$ when $\eta = 0.25$ and $\kappa = 2, 3$, respectively, and $0.0026, 0.0043$ when $\eta = 0.50$ and $\kappa = 2, 3$, respectively. (Note that this means that when applying simply the normal chart for these distributions, FAR is 3.6 to 9.3 times as large as it should be!) For the Gamma(4, 1)-distribution we get $ME = 0.0080$.

3.1 Normal distribution

The upper limit for AVE is directly obtained from (2), since under IC $m^{1/2}\overline{Y}$ has a $N(0,1)$-distribution and hence $UL(AVE, m) = u_{mp}$. We take the shift d according to (1) and hence the ARL of IND runs from 10 to 200. For the normal distribution the ARL under OoC of AVE is given by

$$ARL(AVE, m, d) = \frac{m}{\overline{\Phi}\left(u_{mp} - m^{1/2}d\right)}. \tag{9}$$

In Figure 1 the difference between the group chart and the individual chart is shown.

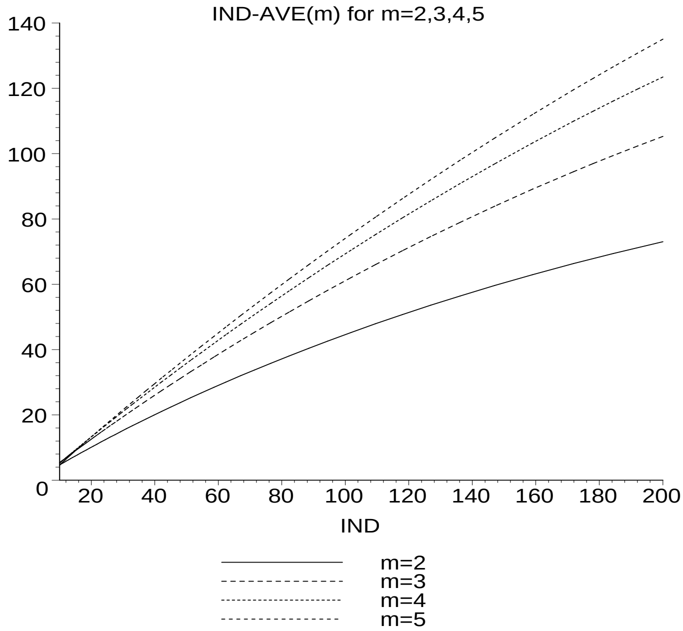

Figure 1. Averagechart under normality.

For a large ARL of IND, that is for small d, a larger value of m gives a nonnegligible improvement, but for a shift $d \geq 1$ (corresponding to $ARLIND \leq 55$) the differences between $m = 3, 4, 5$ are not very large. From (9) it is evident that the individual chart will beat the m-chart ($m \geq 2$) definitely as soon as $1/\overline{\Phi}\,(u_p - d) \leq m$, i.e. when $d \geq \widetilde{d_1} = u_p - u_{1/m}$. For $p = 0.001$, we obtain $\widetilde{d_1} = 3.09, 2.66, 2.42$ and 2.25 for $m = 2, 3, 4$ and 5, respectively. Note that these values are quite close to the actual d_1 for which $ARL(AVE, m, d) = ARL(AVE, 1, d) : d_1 = 2.97, 2.63, 2.40$ and 2.24 for $m = 2, 3, 4$ and 5, respectively. Hence, the individual chart beats the various AVE's, but not really much sooner than in the obvious case where $ARL(AVE, 1, d) \leq m$.

Next we consider MIN for $m = 1, ..., 5$.

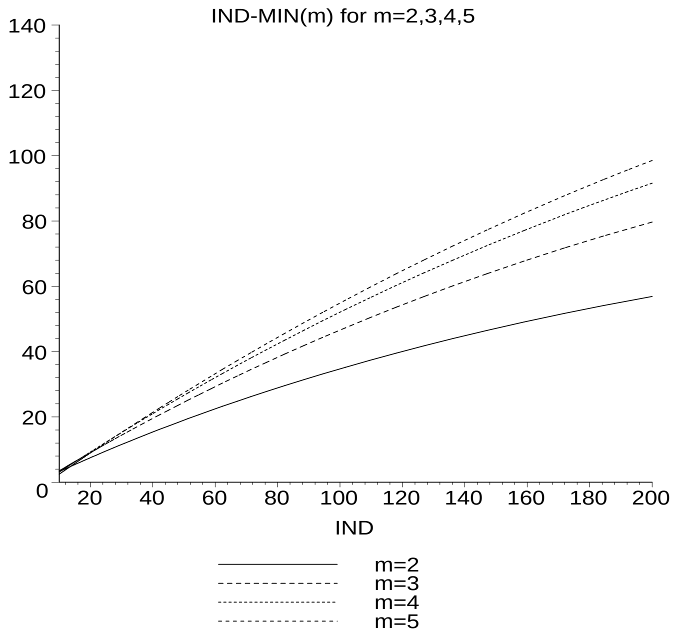

Figure 2. Minimumchart under normality.

The same pattern is seen as in Figure 1: for small d a substantial gain can be obtained when using larger values of m, but for $d \geq 1$ the differences between $m = 3, 4, 5$ are small. The values of d for which the individual chart is better are somewhat lower than when dealing with AVE, but still they are so large, that no important gain can be obtained. Figures 1 and 2 together show that even for normally distributed observations MIN actually performs quite well, in particular if we compare it with the individual chart. For example, at $d = 1$ the ARL of the individual chart equals 54.6; it is improved with 26.7 by taking MIN with $m = 3$, yielding $ARL = 27.9$; the further improvement when using AVE with $m = 3$ is much less: 8.5, giving $ARL = 19.4$.

As stated before, MAX is very close to the individual chart. In fact, it is even slightly worse: for $0.5 \leq d \leq 2$ and $2 \leq m \leq 5$ it turns out that $ARL(MAX, 1, d) - ARL(MAX, m, d)$ lies between -2.3 and -0.4. Hence, MAX is no serious option.

Taking MIX with $\gamma = 1 - (4m)^{-1}$, a slight improvement w.r.t. MIN is obtained as is seen in Figure 3. For instance, the difference between MIX and MIN at $d = 0.8$, where the individual chart has $ARL = 91$, goes from 5.1 for $m = 2$ to 6.5 for $m = 5$. Therefore, it seems that this type of improvement does not outweigh the increased complexity of the resulting chart.

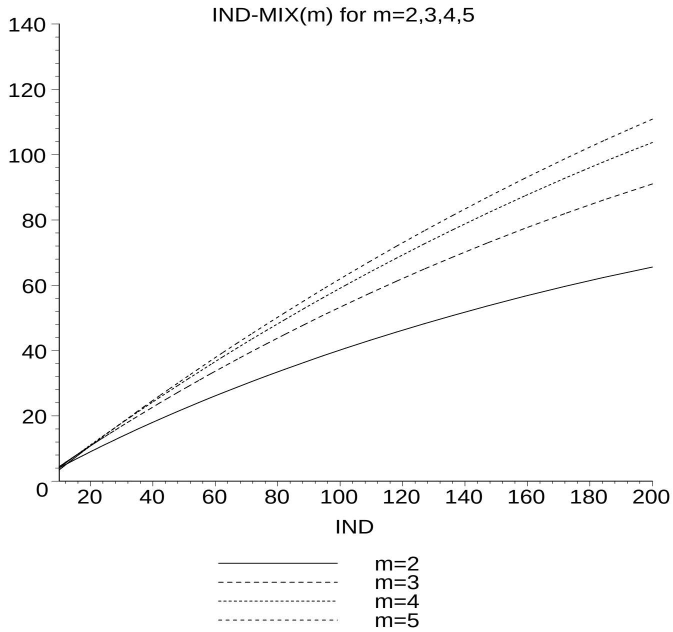

Figure 3. MIX-chart under normality.

Next we consider the UNI-chart. The probability of a signal for this case equals

$$P(\sum_{i=1}^{m} \left\{ \Phi(Y_i^{(0)} + d) \right\} > m - \{m!(mp)\}^{1/m})$$

with $Y_1^{(0)}, ..., Y_m^{(0)}$ a sample from the standard normal distribution, cf. also (4). The resulting expressions are much less explicit than the corresponding ones for AVE and MIN. Hence in this sense UNI is somewhat less attractive to work with. Moreover, for larger m we are more dealing with the shift $E\Phi(Y_i^{(0)} + d) - E\Phi(Y_i^{(0)})$ than with the shift d of Y_i itself. Figure 4 shows the behavior for AVE, MIN and UNI when $m = 2$. It is seen that UNI appears to lie between AVE and MIN: it also loses a bit compared to the optimal AVE, but even less than MIN. This agrees with the intuition according to which $\Phi(Y_1) + \Phi(Y_2)$ is somewhat closer to $Y_1 + Y_2$ than $min(Y_1, Y_2)$.

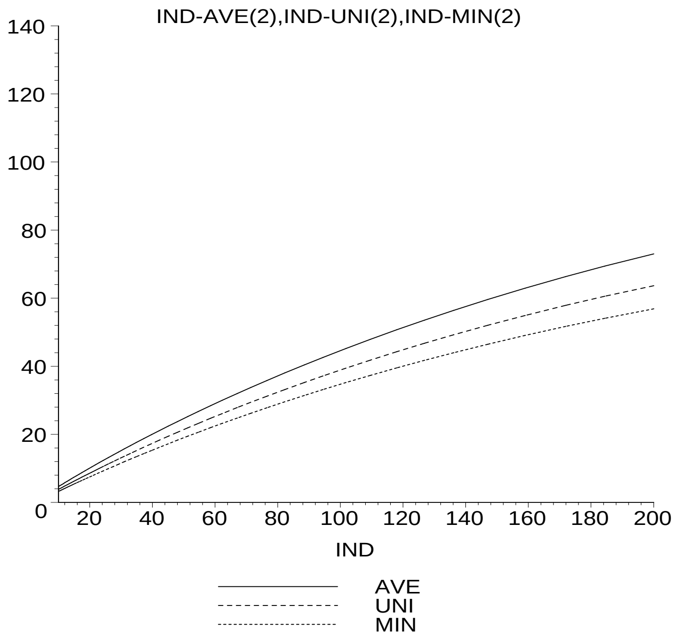

Figure 4. Difference between individual chart and $m = 2$-chart for AVE,UNI and MIN.

To conclude this section, we summarize the above by means of an explicit example. Remember that $p = 0.001$. Hence for $m = 2$ we subsequently have that a signal occurs for

- the individual chart if Y_1 exceeds $u_{0.001} = 3.09$ and otherwise (or again) if $Y_2 \geq 3.09$,
- AVE if $Y_1 + Y_2$ exceeds $u_{0.002}\sqrt{2} = 4.07$,
- MIN if both Y_1 and Y_2 exceed $u_{0.045} = 1.70$,
- MAX if Y_1 and/or Y_2 exceeds $u_{0.001} = 3.09$,
- MIX (with $\gamma = 1 - 1/8$) if both if both Y_1 and Y_2 exceed $u_{0.092} = 1.33$ and at least one of these exceeds $u_{0.012} = 2.27$,
- UNI if $\Phi(Y_1) + \Phi(Y_2)$ exceeds $2 - (0.004)^{1/2} = 1.94$.

In view of the results under normality we restrict ourselves for the remaining distributions to AVE and MIN. We do not further consider UNI for two reasons: firstly, because it seems to be intermediate between AVE and MIN. Secondly, the ARL of UNI is quite complicated; straightforward application of Maple results in serious numerical problems.

3.2 Random Normal Mixture

In this section we consider the random normal mixture given by

$$F(x) = (1 - \eta)\Phi\left(\frac{x}{\sigma_1}\right) + \eta\Phi\left(\frac{x}{\sigma_2}\right),$$

where the σ_i are such that the variance equals 1, that is $(1 - \eta)\sigma_1^2 + \eta\sigma_2^2 = 1$. Obviously, these distributions are still symmetric around 0, but the tail behavior differs from that of the normal distribution. The normal df occurs when $\kappa = \sigma_2/\sigma_1 = 1$. We will consider here $\kappa = 2, 3$ together with $\eta = 0.25$ and 0.50. Figures $5 - 8$ show the differences between the individual chart ($m = 1$) and the average- and minimum-chart with $m = 2$ under the normal mixtures. For $\eta = 0.25$ the minimum-chart is superior to the average-chart both for $\kappa = 2$ and 3. This illustrates that the superiority of AVE in the normal case – where it is simply optimal – can indeed be lost if κ moves away from 1. Since we are dealing with the very far tails, for rather unbalanced variances σ_1^2 and σ_2^2 the normal distribution with the larger variance will soon dominate in the mixture. As a consequence the tail behavior of the mixture will be close to that of the normal distribution. For larger values of η like 0.75 and $\kappa = 2, 3$ (not presented here) this is clearly seen. Indeed, in Figures 7 and 8 AVE starts to beat MIN again.

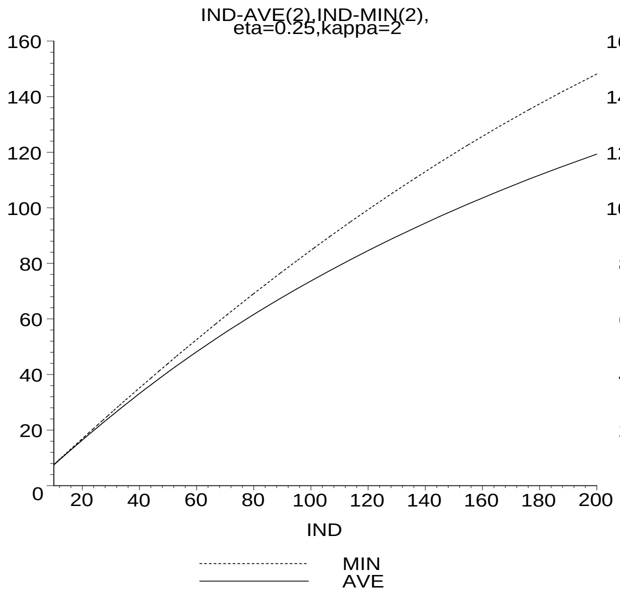

Figure 5. AVE and MIN under normal mixture with $\eta = 0.25$ and $\varkappa = 2$.

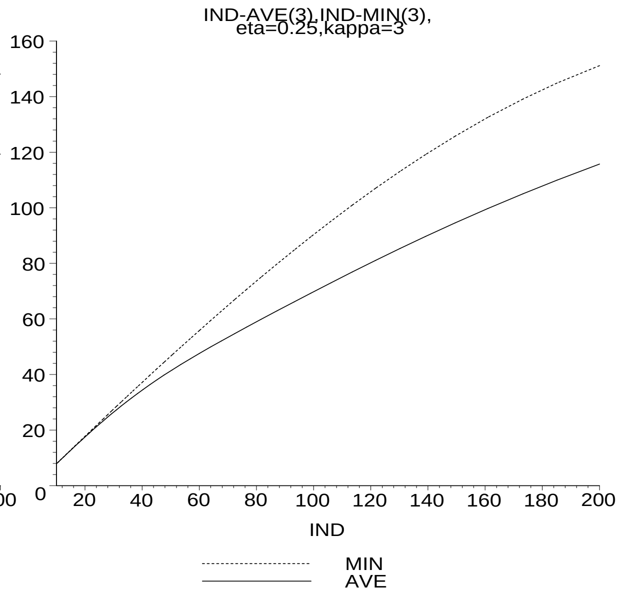

Figure 6. AVE and MIN under normal mixture with $\eta = 0.25$ and $\varkappa = 3$.

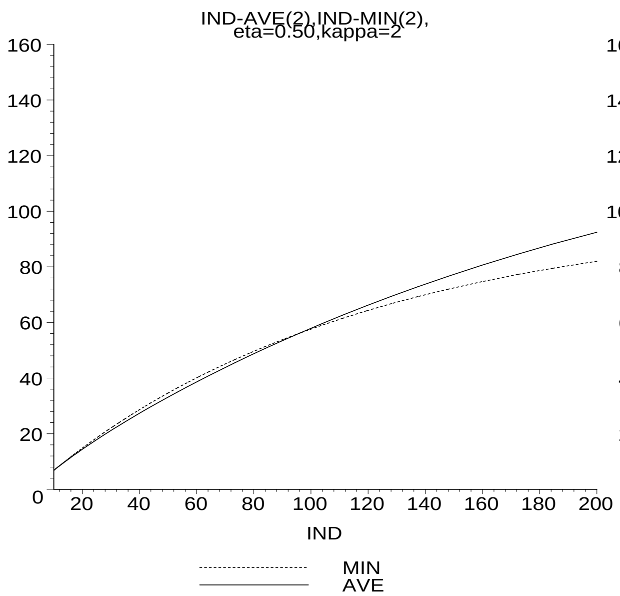

Figure 7. AVE and MIN under normal mixture with $\eta = 0.50$ and $\varkappa = 2$.

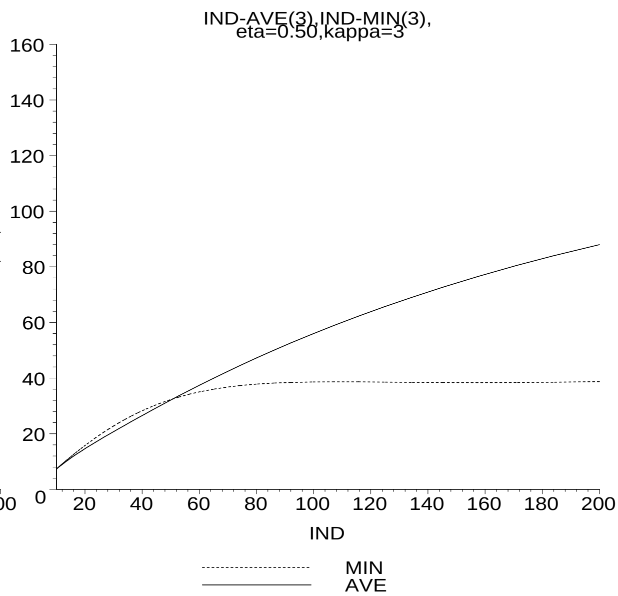

Figure 8. AVE and MIN under normal mixture with $\eta = 0.50$ and $\varkappa = 3$.

3.3 Gamma

Here we consider as an example of a skew distribution the Gamma distribution with parameters 4 and 1 having density $\frac{1}{6}x^3e^{-x}$. Its coefficient of skewness equals 1. In Figure 9 the difference of the ARL's of AVE and MIN are plotted against the ARL of AVE.

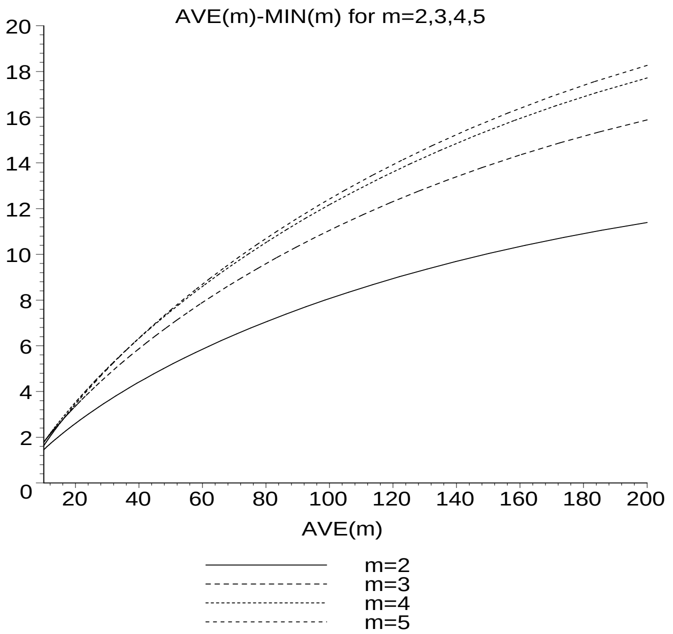

Figure 9. Difference between the ARL's of AVE and MIN

It is seen that MIN is somewhat better than AVE. Both of them are much better than the individual chart. For instance the ARL of the individual chart at $d = 1$ equals 213.2, the ARL of the MIN-chart at $d = 1$ equals 79.6, 41.1, 26.2, 19.3 for $m = 2, 3, 4, 5$, respectively, while the AVE-chart at $d = 1$ gives 87.1, 47.8, 31.4, 23.3 for $m = 2, 3, 4, 5$, respectively.

4 Nonparametric Estimation

Nonparametric estimation of the control limit goes well beyond the intention of this paper, but here we give a rough sketch of the intended approach, as it is important with respect to judging the benefits of the control charts AVE, UNI and MIN.

We start with AVE. Here we have to estimate the $(mp)^{th}$-quantile of the mean of $Y_1, ..., Y_m$. The easiest way of handling is to estimate the mean and standard deviation of the Y_i's and to apply the normal chart to the standardized observations. The argument is that the standardized mean of $Y_1, ..., Y_m$ can be approximated well by the normal distribution. Moreover, a slightly less extreme quantile than for the individual chart is required. The larger m, the better this should work. Unfortunately, this is for $m = 3$ and even for $m = 5$ often too optimistic. We illustrate this by the examples of Section 3. First, we take $m = 3$. The 0.003-quantile of the standard normal distribution equals 2.748. For the random normal mixture we get $P(3^{1/2}\overline{Y} > 2.748) = 0.0053, 0.0045$ for $\kappa = 2$ and $\eta = 0.25, 0.50$, respectively and $0.0077, 0.0055$ for $\kappa = 3$ and $\eta = 0.25, 0.50$, respectively. For the Gamma distribution with parameters 4 and 1 we get $P(3^{1/2}\overline{Y} > 2.748) = 0.0099$, which even is 3.3 times as large as it should be! Next consider $m = 5$. The 0.005-quantile of the standard normal distribution equals 2.576. For the random normal mixture we get $P(5^{1/2}\overline{Y} > 2.576) = 0.0067, 0.0062$ for $\kappa = 2$ and $\eta = 0.25, 0.50$, respectively and $0.0086, 0.0070$ for $\kappa = 3$ and $\eta = 0.25, 0.50$, respectively. For the Gamma distribution with parameters 4 and 1 we get $P(5^{1/2}\overline{Y} > 2.576) = 0.0115$, which is 2.3 times as large as it should be.

A possible approach to estimating the $(mp)^{th}$-quantile of the mean of $Y_1, ..., Y_m$ in a nonparametric way is to use the $\binom{n}{m}$ groups of m observations obtained from $X_1, ..., X_n$ and to take a suitable order statistic among these $\binom{n}{m}$ "observations". It seems that there is at present not much known about the performance of such estimators for moderate n and very small p.

In the uniform-chart we estimate F by the empirical df of $X_1, ..., X_n$ and get a sum of ranks based statistic. Although not that easy, one can deal with it for $m = 2, ..., 5$.

By far the easiest case is the minimum-chart. Estimation of the p^{th}-quantile of F is replaced by estimation of the $\left(\{mp\}^{1/m}\right)^{th}$-quantile. This can be done straightforwardly by the corresponding sample quantile, since we are no longer dealing with the extreme tail of F.

5 Conclusions

The results of this paper lead to the following conclusions.

1. The chart based on a group of $m = 2, ..., 5$ in general performs better than the individual chart.

2. The charts based on the average or on the minimum of the group of observations together with the so called uniform-chart, are the most promising ones.

3. Application of the average-chart as a normal chart may lead to large relative errors under IC.

4. Accurate nonparametric estimation for the minimum-chart is quite straightforward: instead of estimation of a very extreme quantile a rather modest quantile of the distribution has to be estimated. Nonparametric estimation for the average-chart and the uniform-chart seems to be possible, but is far more complicated.

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